Whatever a great man does, the same is done by others as well. Whatever standard he sets, the world follows."

This is the philosophy, on which my venture to write this book is based. No doubt, in many an occasion I have borrowed the ideas developed by the pioneer workers, nevertheless you will find some original ideas in the pages.

While a wealth of information exists on the subject of phase locking and while about half a dozen of textbooks have been written on phase locked loops perhaps there has not appeared a book that has laid stress on the importance of the principle of direct synchronization in automatic phase control circuit. Another departure from the convention followed by most books on synchronization is the inclusion of Chapters on the various application-aspects of the principle of direct synchronization, digital phase locked loops and a short survey on the applications of synchronization through electrical and optical terminals of the devices.

The book really consists of five inter-related parts. The first part incorporates Chapter-I, which gives a sort of review of fundamentals necessary for building a study on the operation of auto-
Chapters 1 through 5, describe phase-locked loops. While Chapter 6 explains the design of phase-locked circuits. The second part of the book concentrates on the operational aspects of phase-locked loops. Finally, Chapter 11 overviews of phase lock techniques. Chapter 12 is as follows:

Chapter 12 presents phase-locked loop filters and operational amplifiers. This chapter was introduced in Chapter 5. Chapter 12 explains the filter design using operational amplifiers. Techniques for designing operational amplifiers will be also discussed. Chapter 13 describes the behavior of a phase-locked loop. The behavior of a phase-locked loop has been determined in this chapter. Chapter 14 deals with the design of operational amplifiers. Techniques for designing operational amplifiers will be also discussed. Chapter 15 describes the behavior of a phase-locked loop in connection with the design of a high-order system. Chapter 16 deals with the design of a phase-locked loop in connection with the design of a high-order system.
detail the phase detector response characteristics in response to various types of noisy signals. In Chapter-15 nonlinear analysis of phase locked loops via Fokker-Planck and quasi-linearization techniques has been carried out.

Chapter-16 presents a brief study on the behaviour of digital phase locked loops to noise-free and noisy signals. Finally, Chapter-17 presents a short survey on the application of phase lock principles extending up to microwave and millimetre wave frequencies using the devices' electrical and optical terminologies.
THE BIRTH OF PHASE LOCKING

The origin of phase locking or automatic phase control dates back to the seventeenth century, when Huygens (1629-1695) reported his famous experiment on clock-synchronization in 1665, an imaginative model of which is shown in the given figure. He found that two clocks (pendulums), which were initially slightly out-of-step with each other, when fixed on a thick wooden board, became synchronized with each other when hung on a thin wooden board.

However, the significance of Huygens' discovery could not be appreciated till the dawn of the twentieth century when Appert and van der Pol in 1922 re-discovered the phenomenon in electronic circuits. At this stage, the birth of the most elemental form of the phase-lock circuit or automatic phase control circuit took place.
The Birth of Phase Locking

And the device that achieved synchronization, is said to be the directly or in series synchronized oscillator. Since then the phenomenon has continued to trigger interest of many physicists and engineers, and has ultimately led to the discovery of the present form of the automatic phase control circuits, known as the phase-locked loop or phase-locked oscillator, apparently first described by de Boolec in 1932.
ACKNOWLEDGEMENTS

I am profoundly grateful to my teacher, Professor NB Chakraborty of Indian Institute of Technology, Kharagpur, India, whose guidance and blessings the treasure-island of non-linear conditions would have remained undiscovered by me.

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Barishan, India

R.N. DRIWAS
CONTENTS

Preface 31
Acknowledgements 37
The Birth of Phase Locking 38

Chapter 1. FUNDAMENTALS 3
1.1. Fourier Transform 3
1.2. Laplace Transform 3
1.3. Network Response 3
1.4. Linear Feedback System 9
1.5. Stability Analysis 10
1.6. Random Signal Theory 11
1.7. Optimum Linear System 19
1.8. White Noise and Narrowband Noise 22
1.9. Noise bandwidth 29

Chapter 2. NEARLY SINUSOIDAL OSCILLATIONS 28
2.1. Introduction 33
2.2. Sinusoidal Oscillations in Linear Systems 33
2.3. Phase Plane or State Space Approach 37
2.4. Nearly Sinusoidal Oscillations 44
2.5. Solutions of the Oscillator Equations 46
2.6. Phase Plane Notes and the van der Pol's Equations 50
2.7. Asymptotic Techniques 54
2.8. Application of the Asymptotic Method 57
2.9. The Solution of van der Pol's Equation 58
2.10. Stationary Oscillations and Its Stability 68
2.11. SoC Self-excited Oscillations 60
2.12. Hard Self-excited Oscillations 65
2.13. Remarks 68
Chapter 3. INFLUENCE OF A SINEWOBAL SIGNAL ON CLASS-A OSCILLATIONS

3.1. Introduction 76
3.2. Analytical Method 83
3.3. Entrainment of a van der Pol Oscillator 85
3.4. Spectral Character of Unlocked
      Determinant Oscillator 97
3.5. Remarks 99

Chapter 4. RESPONSE TO NOISE-FREE MODULATED SIGNALS 102

4.1. Introduction 102
4.2. Response to an AM Signal 102
4.3. Modulating Frequency Smaller than the Synchronization Range 103
4.4. Response to an FM Signal 107
4.5. Remarks 117

Chapter 5. RESPONSE TO NOISY CW SIGNALS 119

5.1. Introduction 119
5.2. Derivation of System Equations 119
5.3. Weak Interference Lying Away from the
      Oscillator Centre Frequency 121
5.4. Strong Interference 124
5.5. Signal Contaminated with Gaussian Noise 125
5.6. Remarks 138

Chapter 6. FILTERING AND AMPLIFYING PROPERTY 141

6.1. Introduction 141
6.2. Locking Bandwidth for a Noise-free FM Signal 141
6.3. Synchronization for a Noisy FM Signal 143
6.4. Filtering Property 148
6.5. Amplifying Property 152
6.6. Remarks 155

Chapter 7. PHASE-LOCKED LOOP FUNDAMENTALS 157

7.1. Mechanization of the Phase-Locked Loop 157
Chapter 8. NOISE-FREE ANALYSIS OF LINEARISED LOOPS

8.1. Linearized Loop Equation 187
8.2. Loop Transfer Functions and Root Locus Plots 186
8.3. Trajactory Response 188
8.4. Response to an Angle Modulated Signal 201
8.5. Nonlinear Operation of Phase Locked Demodulators 205
8.6. Reception of a Doppler Shifted FM Signal 213

Chapter 9. LINEAR ANALYSIS WITH STOCHASTIC INPUTS

9.1. Characterization of Input Noise 219
9.2. Derivation of the Loop Equation in the Presence of Additive Noise 220
9.3. Mean Square Phase Error and Noise Bandwidth 222
9.4. Optimization of Loop Performance 229
9.5. Loop Parameters Selection in Practice for Frequency and Phase Offset 235
9.6. PLL Demodulator and Standard Limiter Discriminator 237
9.7. Optimum Phase Locked Demonstration of an FM Signal 243

Chapter 10. NONLINEAR BEHAVIOUR OF NOISE-FREE LOOPS

10.1. Signal Acquisition by a Second Order, Type-1 and Type-2 Loops 260
10.2. Approximate Acquisition Analysis of a Second Order, Type-2 Loop 268
10.3. Approximate Acquisition Analysis of the PLL Incorporating an Imperfect Integrator 272
10.4. Approximate Acquisition Analysis of a Second Order Type-2 Loop 275
10.5. Unified Approach for Acquisition Analysis of a PLL Incorporating Filter Networks with and without High Frequency Gain 278
10.6. Locking Characteristics of a PLL with Triangular, Sawtooth and Rectangular Type of Phase Detectors 282
10.7. Simplified Formula for an Arbitrary Periodic Phase Detector 288

Chapter 11. RANGE EXTENSION
11.1. Extended Range Phase Locked Loop 292
11.2. Frequency Feedback Phase Locked Loop 295
11.3. Injection Synchronized Phase Locked Loop 299
11.4. Test-tone Modulated Signal Fed to the Modified Loops 303
11.5. Forced Signal Acquisition 305

Chapter 12. HETERODYNE AND MULTIFILTER LOOPS 316
12.1. The Heterodyne Phase Locked Loop 310
12.2. Nonlinear Behaviour 322
12.3. Stability 325
12.4. Fase Acquisition 328
12.5. Multifilter PLL 333
12.6. Multiloop Multifilter PLL 340

Chapter 13. USE OF A BANDPASS LIMITER 345
13.1. Transform Method of Analysis for a BPL 345
13.2. PLL Precise by a BPL 357

Chapter 14. PHASE DETECTOR RESPONSE TO NOISY SIGNALS 362
14.1. Realization of Various Types of Phase Detectors 362
14.2. Signal Plus AWGN to PD Input 368
14.3. Noise Analysis of Phase Detectors 371
14.4. Remarks on the Phase Detector Gain and Threshold Properties 375
14.5. Response of a PD to Noisy Fading Signals 376
Chapter 15. NONLINEAR ANALYSIS WITH NOISY SIGNALS

15.1. Random Walk Problem and Markovian Process 387
15.2. Smoluchowski Equation 360
15.3. Voicier-Planck Equation 391
15.4. Noise Performance of a First Order PLL 294
15.5. Cycle Slipping Phenomenon 464
15.6. Nonlinear Analysis of Second Order Loop 496
15.7. Quadrature Technique 472
15.8. Acquisition Analysis in a Noisy Environment 673
15.9. Response of a PLL to a Test-tone Modulated Signal Coprocessed with AWGN 416
15.10. Phase Locked Discriminator Performance 424

Chapter 16. DIGITAL PHASE-LOCKED LOOPS 433

16.1. Introduction 433
16.2. All Digital Phase-Locked Loop 437
16.3. Response of a First Order DPLL to a Frequency Step Input 437
16.4. Digital Phase-Locked Loop 439
16.5. Derivation of the System Equation 441
16.6. First Order DPLL with Phase-step Input 444
16.7. First Order DPLL with Frequency Step or Phase Ramp Signal 446
16.8. Response of the Second Order Loop to a Frequency Step Signal 446
16.9. Response to a Frequency Ramp Signal or a Doppler Rate Input 452
16.10. Extended Range DPLL 453
16.11. DPLL Response to Noisy Signal 454
16.12. Scrambled Order DPLL and Filter Design 462
16.13. Remarks 467

Appendix A. 469

FURTHER APPLICATIONS OF PHASE LOCK PRINCIPLES 476

1. Oscillator Cleaning 478
17.2. Demodulation 480
17.3. FM/PM Modulation 482
17.4. Filtering 483
17.5. Colour Identification in Television 484
17.6. Frequency Synthesizer 485
17.7. mm-Wave Pulsed Coherent Signal 486
17.8. Phased Array Antenna 487
17.9. Digital Phase Shifting 488

Glossary of Symbols  491
Index  496
CHAPTER 1

FUNDAMENTALS

Understanding of phase lock theories requires a knowledge of several disciplines of the engineering mathematics, the nonlinear mechanics and control theory. The purpose of this Chapter is, therefore, to refresh the memory of the reader with appropriate concepts, formula and language of network analysis, feedback theory and stochastic analysis. In this Chapter we will be concerned with the fundamentals of linear analysis and the concepts of non-linear analysis, such as, the quasi-linear technique, Fokker-Planck approximation, etc., will be introduced in the appropriate Chapters. Obviously, there is no scope for introducing these materials in detail and in such the reader may refer to the bibliography at the end of this Chapter for further elaboration and clarification.

1.1 Fourier Transform

A function \( x(t) \) in the time domain can be transformed to another function \( X(f) \) in the frequency domain with the help of the Fourier transform, which is defined as

\[
X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) \, dt
\]

where

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) \exp(j2\pi ft) \, df
\]  

(1.1)  

(1.2)

Note that \( x(t) \) is Fourier transformable provided (1) the function \( x(t) \) is single valued, (2) has finite number of maxima and minima,
2. **Point Lock Theories and Applications**

(a) has a finite number of discontinuities in any finite interval, and
(b) $x(t)$ is absolutely integrable, i.e.,

$$\int_{a}^{b} |x(t)| \, dt < \infty$$

Properties of the Fourier transform are summarized in Table 1.1.

### Table 1.1

<table>
<thead>
<tr>
<th>Property</th>
<th>Mathematical representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Superposition</td>
<td>$F(aX+bY(t)) = e^{j\omega_0 t} X(f)$ where $a$ and $b$ are constants</td>
</tr>
<tr>
<td>2. Time shifting</td>
<td>$F(x(t-T)) = X(f) e^{-j\omega_0 T}$</td>
</tr>
<tr>
<td>3. Time scaling</td>
<td>$F(aX(t)) = \frac{1}{a} X(f/a)$</td>
</tr>
<tr>
<td>4. Frequency shifting</td>
<td>$F(x(t) e^{j2\pi ft}) = X(f-\omega_0)$</td>
</tr>
<tr>
<td>5. Duality</td>
<td>If $F(x(t)) = X(f)$ then $F(X(f)) = x(-t)$</td>
</tr>
<tr>
<td>6. Area under $x(t)$</td>
<td>$\int_{-\infty}^{\infty} x(t) , dt = 0$</td>
</tr>
<tr>
<td>7. Area under $X(f)$</td>
<td>$\int_{-\infty}^{\infty} X(f) , df = 0$</td>
</tr>
<tr>
<td>8. Differentiation</td>
<td>$F(\frac{d}{dt} x(t)) = j2\pi f X(f)$</td>
</tr>
</tbody>
</table>
Fundamentals 3

<table>
<thead>
<tr>
<th>Property</th>
<th>Mathematical representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>9. <strong>Integration</strong></td>
<td>[ F \left[ \int x(t) , dt \right] = \frac{1}{j2\pi f} \mathcal{X}(f) + \frac{\mathcal{X}(0)}{2} i f ]</td>
</tr>
<tr>
<td>10. <strong>Multiplication</strong></td>
<td>[ F \left[ x(t) y(t) \right] = \int_x \mathcal{X}(f) , y(f - \omega) , df ]</td>
</tr>
<tr>
<td>11. <strong>Convolution</strong></td>
<td>[ F \left[ \mathcal{L} { h(t) * x(t) } \right] = \mathcal{X}(f) \mathcal{H}(f) ]</td>
</tr>
<tr>
<td>12. <strong>Conjugate function</strong></td>
<td>[ F { x^<em>(t) } = 2^</em> { -f }, \text{ where } ^* \text{ denotes complex conjugate operation.} ]</td>
</tr>
</tbody>
</table>

1.2 **Laplace Transform**

A function \( x(t) \) in the time domain is transformed to \( X(s) \) in the complex frequency domain \( s = \sigma + js \) with the help of the Laplace transform defined as

\[
L \{ x(t) \} = X(s) = \int_{0}^{\infty} x(t) \exp(-st) \, dt, \text{ Re } s > 0 \tag{1.3}
\]

\[
L^* \{ X(s) \} = x(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} X(s) \exp(st) \, ds \tag{1.4}
\]

where \( c \) is a constant forming a convergent path of integration. The above notation in terms of the unilateral Laplace transform is commonly used for the purpose of transient analysis of a system with signals existing for positive time only. In such a case, the unilateral definition of the Laplace transform becomes identical with the bilateral definition of the Laplace transform where the lower limit of integration of (1.3) is taken to be \( t = -\infty \), i.e.,
In most cases, particularly in our cases, it will be possible to obtain the Fourier transform from the Laplace transform by changing $s$ to $js$, i.e.,
\[ X(js) = X(f) \bigg|_{f=s} \]  
(1.6)
The properties of the Laplace transforms are shown in Table 1.2.

### Table 1.2
**Properties of the Laplace Transform**

<table>
<thead>
<tr>
<th>Property</th>
<th>Mathematical representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Superposition</td>
<td>( L {a x(t) + b y(t)} = a X(s) + b Y(s) ), where ( a ) and ( b ) are constants</td>
</tr>
<tr>
<td>2. Time scaling</td>
<td>( L {x(at)} = \frac{1}{a} X(s/a) )</td>
</tr>
<tr>
<td>3. Time shifting</td>
<td>( L {x(t - \theta)} = X(s)e^{-\theta s} )</td>
</tr>
<tr>
<td>4. Differentiation</td>
<td>( L {\frac{dx(t)}{dt}} = sX(s) - x(0) )</td>
</tr>
<tr>
<td>5. Integration</td>
<td>( L \int_{-\infty}^{t} x(\tau) d\tau = \frac{X(s)}{s} + \int_{-\infty}^{0} x(\tau) e^{\tau s} d\tau )</td>
</tr>
<tr>
<td>6. Initial value</td>
<td>( \lim_{t \to 0} x(t) = \lim_{s \to \infty} sX(s) )</td>
</tr>
<tr>
<td>7. Final value</td>
<td>( \lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) )</td>
</tr>
<tr>
<td>8. Damping</td>
<td>( L {\exp(-at) x(t)} = X(s + a) )</td>
</tr>
</tbody>
</table>
9. Multiplication

\[ L (x(t)) y(t) = \int x(t - \alpha) y(\alpha) d\alpha \]

10. Convolution

\[ x(t) * y(t) = \int x(\tau) y(t - \tau) d\tau \]

\[ = \int y(t - \tau) x(\tau) d\tau \]

### 1.3 Network Response

Let us consider the two-terminal time-invariant linear network of Fig. 1.1. For an input \( x(t) \) let the network produce an output \( y(t) \).

**Fig. 1.1.** The time-invariant linear system.

Obviously, an input \( x(t + \alpha) \) will also produce an output \( y(t + \alpha) \). For all linear time-invariant networks, the output and the input may be related by a set of linear constant-coefficient differential equations, which when solved for the output give the time response of the network. The time response of the network can also be computed in another way, which we will indicate a little later.

For the moment we consider the frequency response of the network. For that matter, let the input be of the form \( \exp(j\omega t) \), then the output will be also of the form \( \exp(j\omega t) \), because the linear system cannot generate new frequencies. Thus we write
\[ X(j\omega) = \int x(t) \exp(-j\omega t) \, dt \quad (1.7) \]

and

\[ Y(j\omega) = \int y(t) \exp(-j\omega t) \, dt \quad (1.8) \]

The ratio of the output \( Y(j\omega) \) to the input \( X(j\omega) \) is termed as the transfer function of the system, \( H(j\omega) \), i.e.,

\[ H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \quad (1.9) \]

Let us consider as important case when the input \( x(t) \) is a unit impulse, i.e.,

\[ x(t) = \delta(t) \quad (1.10) \]

where \( \delta(t) \) is the Dirac's delta function. Then we get

\[ X(j\omega) = \int x(t) \exp(-j\omega t) \, dt \]

which by definition of the Dirac's delta function turns out to be

\[ X(j\omega) = 1 \quad (1.11) \]

The output obtained under the influence of the unit impulse, is called the weighting function or the impulse response of the network. From (1.9) and (1.11) one finds that

\[ Y(j\omega) = H(j\omega) \]

Hence

\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega) \exp(j\omega t) \, d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) \exp(j\omega t) \, d\omega \quad (1.12) \]

The output obtained under this condition is denoted by \( h(t) \), which indicates the weighting function or the impulse response of the network. That is,
\[ h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) \exp(j \omega t) \, d\omega \quad (1.12) \]

and conversely,
\[ H(j\omega) = \int_{-\infty}^{\infty} h(t) \exp(-j\omega t) \, dt \quad (1.14) \]

We can write the output in terms of the input and the impulse response of the network by using
\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega) \exp(j \omega t) \, d\omega \quad (1.9) \]

and (1.9). That is,
\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) X(j\omega) \exp(j \omega t) \, d\omega \quad (1.15) \]

Using (1.14) is (1.15) we write
\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \exp(-j\omega u) X(j\omega) \exp(j \omega t) \, d\omega \, du \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \exp(j \omega t-\omega u) \, d\omega \right] \, du \quad (1.16) \]

Using the time shifting property of the Fourier transform, we find from (1.16)
\[ y(t) = \int h(u) x(t-u) \, du \quad (1.17) \]

or
\[ y(t) = \int x(t-u) h(t-u) \, du \quad (1.18) \]

The above relationships are commonly known as convolution inte-
Phase Lock Theories and Applications

guals. Taking the Laplace transform of (1.17) or (1.18) we find,

$$\mathcal{Y}(s) = H(s) \mathcal{X}(s)$$  \hspace{1cm} (1.19)

Again coming back to the frequency response function $H(j\omega)$, we note that it is, in general, a complex quantity. That is, it can be expressed as

$$H(j\omega) = A(\omega) \exp(j\phi(\omega))$$  \hspace{1cm} (1.20)

where $A(\omega)$ is called the amplitude response and $\phi(\omega)$ is called the phase angle. In the case of linear time invariant systems, $A(\omega)$ is an even function of $\omega$ and $\phi(\omega)$ is an odd function of $\omega$. In certain cases of time invariant linear networks, known as minimum phase networks, which includes most of our common networks with some exceptions, such as lattice type, the gain and phase components are related as

$$\phi(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(\omega_0)}{\omega - \omega_0} \, d\omega_0$$  \hspace{1cm} (1.21)

and

$$a(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A(\omega_0)}{\omega - \omega_0} \, d\omega_0$$  \hspace{1cm} (1.22)

where

$$a(\omega) = \ln[A(\omega)]$$  \hspace{1cm} (1.23)

since $a(\omega)$ is an even function of $\omega$ and $\phi(\omega)$ is an odd function of $\omega$, (1.21) and (1.22) can be written as

$$\phi(\omega) = \frac{2\pi}{\pi} \int_{-\infty}^{\infty} \frac{\ln[A(\omega_0)]}{\omega - \omega_0} \, d\omega_0$$  \hspace{1cm} (1.24)

and

$$a(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(\omega_0)}{\omega - \omega_0} \, d\omega_0$$  \hspace{1cm} (1.25)
1.4 Linear Feedback Systems

Referring to the block diagram of a typical feedback network, as shown in Fig. 1.2, we write

\[ V(s) = \theta(s) - \theta_d(s) \]
\[ V(s) = A(s) \xi(s) \]
\[ \theta_d(s) = \xi(s) \psi(s) \]

where \( A(s) \) denotes the transfer function of the network in the forward path and \( \psi(s) \) is the transfer function of the network in the feedback path. \( \theta(s) \) and \( \theta_d(s) \) are respectively the input and the output variables, \( \xi(s) \) is the error signal. From (1.26) it is easily shown that the closed-loop transfer function \( F(s) \) and the closed-loop error function are respectively given by

\[ H(s) = \frac{V(s)}{\xi(s)} = \frac{A(s)}{1 + A(s) \psi(s)} \]

and

\[ e(s) = \frac{1}{1 + A(s) \psi(s)} \]

The open loop gain is defined as

\[ G(s) = \frac{\theta(s)}{\theta_d(s)} = A(s) \psi(s) \]

in writing this, a break is made at the point of Fig. 1.2 designated by \( y \). Here \( \theta_d(s) \) denotes the output in this case.
1.5 Stability Analysis

In designing a feedback network, unless it is an oscillator, care is always taken to see that the system does not break into oscillation at any frequency \( \omega \), the system is stable. To check the stability, there are a number of classical analytical tests. Referring to (1.27) one finds that the system response \( H(x) \) becomes unbounded, when

\[
A(j \omega) G(j \omega) = -1
\]

i.e.,

\[
G(j \omega) = -1
\]

That is, we say that system becomes unstable if the open loop amplitude response \( |G(j \omega)| \) is greater than or equal to unity with a phase response of 180°. Referring to the amplitude and phase response relation of (1.24) and utilizing the following integral

\[
\int_{0}^{\infty} \frac{\ln(a)}{s^2 - k \omega^2} ds = -\frac{\pi}{2k \omega} \quad ab > 0
\]

it can be concluded that

(i) when the amplitude response falls by 6dB/Octave, a phase change of 90° occurs (put \( \ln(a) = -1 \)), in (1.24) and use (1.31))

and

(ii) when the amplitude response falls by 12dB/Octave, a phase change of 180° appears (put \( \ln(a) = -2 \)).

Thus a minimum phase network in which the amplitude response has a slope less than 12dB/Octave is unconditionally stable. There is another common way of judging the stability of the system. This depends on the poles' representation of \( H(j \omega) \), which may be represented generally as a ratio of two polynomials in \( \omega \), viz.

\[
\frac{a_0 + a_1 \omega^1 + a_2 \omega^2 + \ldots + a_n \omega^n}{b_0 + b_1 \omega^1 + b_2 \omega^2 + \ldots + b_m \omega^m}
\]

If the input is a unit step, \( u(t) = 1 \), and the output is written as

\[
\frac{a_0 + a_1 \omega^1 + a_2 \omega^2 + \ldots + a_n \omega^n}{b_0 + b_1 \omega^1 + b_2 \omega^2 + \ldots + b_m \omega^m}
\]

which can be written in the following factored form
\[ \mathcal{W}(s) = \frac{A}{s - s_1} + \frac{B}{s - s_2} + \frac{C}{s - s_3} + \ldots \quad (1.24) \]

Note that \(s_1, s_2, s_3\) etc. denote the poles of \(H(s)\), the closed-loop transfer function. Taking the inverse Laplace transforms of (1.24) we write,

\[ \mathcal{Y}(t) = Ae^t + Be^{-t} + Ce^{at} + \ldots \quad (1.35) \]

Referring to (1.35) it is easily concluded that:

(i) When both \(s_1, s_2, s_3\) etc. are real and negative, the output decays in response to an impulse.

(ii) When one or all of the poles \(s_1, s_2, s_3\) etc. are real and positive, the output grows in response to an impulse input.

(iii) When \(s_1\) is pure imaginary, there must be another root \(s\) which is conjugate to \(s_1\). The two roots combine to give sinusoidal solutions. Thus, the output does not decay.

(iv) When \(s_1\) is complex, there must be another root \(s\) which is complex conjugate to \(s_1\). These two roots combine to give either an exponentially decaying sinusoid or an exponentially growing sinusoid depending upon whether the real part of the complex root is negative or positive.

Thus the character of the time function depends entirely on the nature of the poles of \(H(s)\). Obviously, the location of the poles of \(H(s)\) determines the stability of the system and the definition of stability can be stated in the following way:

A system is stable if the closed-loop transfer function has no poles in the right half \(s\)-plane and only simple poles on the \(j\omega\)-axis.

1.6 Random Signal Theory

Basically there are two classes of signals, viz., deterministic signals and random signals. Deterministic signals are those signals which can be denoted by specific functions of time. Whereas, random signals cannot be described by definite functions of time and as such their precise values cannot be predicted in advance. However, they may be characterized by statistical properties. The mathematician discipline that deals with the mathematical description of random signals is known as the probability theory, one of the concepts of which is the probability density function. By referring to Fig. 1.3 which depicts an ensemble of functions \(x_1(t), x_2(t), \ldots, x_d(t)\) constituting...
a random signal $x(t)$, we find that the values assumed by the individual function at any given instant of time $t=t_0$ are random quantities, $x_1(t_0)$, $x_2(t_0)$, ..., $x_d(t_0)$. The probability that $x_n$ lies within a specified range, say $a$ to $b$, is defined by

$$\text{Prob} \ (a < x_n < b) = \int_a^b p(x) \, dx \quad (1.36)$$

we call $p(x)$ the probability density function. A typical variation of $p(x)$ with $x$ is shown in Fig. 1.4. Note that $p(x)$ cannot be negative.
or imaginary and since every measurement must yield some real value, \( p(x) \) must satisfy

\[
p(x)dx = 1
\]

(1.37)

Another important function in the probability theory is called the distribution function \( F(x) \), which is defined as

\[
F(x) = \int_{-\infty}^{x} p(x') dx'
\]

(1.38)

Because of the non-negative character of \( p(x) \), we get

\[
P(-\infty < x < 0) = 0, \quad P(x = \infty) = 1
\]

(1.39)

Incidentally it is worth while to note that among all random processes there is a class called Markov processes. A random process \( x(t) \) is said to be Markovian, if the probabilistic structure of \( x(t) \) for \( t > t_0 \) depends only on the value of \( x(t_0) \) and is independent of all other conditional values of \( x(t) \) for \( t < t_0 \). Moreover, for the Markovian process \( x(t) \), \( x(t_1) - x(t_2) \) and \( x(t_2) - x(t_2) \) are independent random variables whenever \( t_2 > t_1 > t_0 \). One of the statistical characteristics of a random signal is its expected or the average value defined as

\[
\bar{x}(t) = m = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt
\]

(1.40)

We assume that \( x(t) \) is a stationary process, the statistical behaviour of which is independent of the time origin. The mean square value is defined as

\[
\bar{x}^2(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t) dt
\]

(1.41)

and the autocorrelation function is defined as

\[
R(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) x(t + \tau) dt
\]

(1.42)

Incidentally, the random signal \( x(t) \) is said to be a stochastic process...
If the statistical parameters measured over a long interval of time are the same as those computed over the ensemble of states at any given instant of time, then an ergodic process is always stationary whereas a stationary process can be non-ergodic.

The mean or expected value of a random signal $x(t)$ is defined as

$$m = E[x(t)] = E[x(t)] = \int x p(x) \, dx$$

where $E[.]$ denotes expectation. Similarly, the expected value of $f(x)$ is defined as

$$E[f(x)] = \int f(x) p(x) \, dx$$

The $n$th moment of the random variable $x$ is

$$E[x^n] = \int x^n p(x) \, dx$$

The mean square value of $x$ is

$$E[x^2] = \int x^2 p(x) \, dx$$

Central moments are defined as

$$E[(x - m)^n] = \int (x - m)^n p(x) \, dx$$

The variance is thus given by

$$\sigma^2 = E[(x - m)^2] = \int (x - m)^2 p(x) \, dx$$

or

$$\sigma^2 = E[x^2] - m^2$$

The characteristic function $\varphi(v)$ of the probability density function $p(x)$ is defined as

$$\varphi(v) = E[\exp(jv x)]$$
That is,
\[ p(x) = \int_{-\infty}^{\infty} p(y) \exp \left( \frac{-ixy}{\sigma} \right) dy \quad (1.40) \]

The probability density function of a sum of two statistically independent variables \( x \) and \( y \) is given by
\[ p(x) = \int_{-\infty}^{\infty} p(x-y) p(y) dy \quad (1.51) \]

where \( z = x + y \)

Again, it is known that the product \( r = xy \) has the following probability density function
\[ p(r) = \int_{-\infty}^{\infty} p(i) \frac{1}{x} \, dx \quad (1.52) \]

The probability density function \( p(y) \) at the output of a nonlinearity, defined by
\[ y = f(z) \]

can be written in terms of the known probability density function \( p(r) \) by using the following relations
\[ p(y) = \frac{dp}{dy} \quad (1.53) \]

if \( f(z) \) defines a single valued relation between \( y \) and \( x \). On the other hand, if the inverse relation, \( x = g(y) \) is not single valued, then
\[ p(y) = \left[ p(x) \frac{dx}{dy} \right] \frac{dx}{dy} + \left[ \frac{dx}{dy} \right] \frac{dx}{dy} + \ldots \quad (1.54) \]

where \( x_0, x_1, x_2, \ldots \) are the values of \( x \) corresponding to a value of \( y \).

For an ergodic process, the auto-correlation function is defined as
\[ R(t) = E[(x(t) \pi(t + \tau)] \quad (1.55) \]

i.e.,
\[ R(\tau) = \int_0^\infty \int_0^\infty x_1 x_2 p(x_1, x_2) \, dx_1 \, dx_2 \]  
(1.55a)

where \( x_1 \) and \( x_2 \) respectively stand for \( x(t) \) and \( x(t + \tau) \) and \( p(x_1, x_2) \) denotes the two-dimensional probability density function. Similarly, the auto-correlation function at the output of a non-linearity \( y = f(x) \) is given by

\[ R_y(\tau) = \int_0^\infty \int_0^\infty f(x_1) f(x_2) p(x_1, x_2) \, dx_1 \, dx_2 \]  
(1.56)

The power spectral density is related to the auto-correlation function as

\[ S(\omega) = \int_0^{\infty} R(\tau) \, \exp \left( -j \omega \tau \right) \, d\tau \]  
(1.57)

and

\[ R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \, \exp \left( j \omega \tau \right) \, d\omega \]  
(1.58)

There are two other parameters, called the spectral bandwidth \( B_s \) and the correlation time \( \tau_c \) which are sometimes used for a zero mean process. These are defined as

\[ B_s = \frac{1}{\sqrt{\int_0^\infty S(\omega) \, d\omega}} \]  
(1.59)

and

\[ \tau_c = \frac{1}{\sqrt{\int_0^\infty R(\tau) \, d\tau}} \]  
(1.60)

when \( R(\tau) \geq 0 \) for all \( \tau \), then it is shown that,

\[ B_s \, \tau_c = \sqrt{2} \]  
(1.61)

The correlation time \( \tau_c \) gives an idea about the time interval over which correlation is maintained between different samples of \( x(t) \); whereas, the spectral bandwidth denotes an equivalent rectangular power spectral density of bandwidth \( 2B_s \) and of height \( S(0) \) which has the same area as that under the actual power spectral density \( S(\omega) \). Some correlation functions along with their spectral densities,
Correlation functions and spectral densities are given in Table 1.3.

<table>
<thead>
<tr>
<th>Correlation Function</th>
<th>Spectral Densities</th>
<th>Spectral Densitivities</th>
<th>Correlation Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_0 ) ( \exp \left( \frac{-i \gamma}{RC} \right) )</td>
<td>( \frac{N_0 \gamma}{1 + (2\pi i RC)^2} )</td>
<td>( \pi(2\pi RC) )</td>
<td>( \lambda C )</td>
</tr>
<tr>
<td>( N_0 \beta \frac{\sin (2\pi R \gamma) \gamma}{2\pi \beta} )</td>
<td>( \frac{N_0}{2} \exp \left( \frac{-i \gamma}{2\pi} \right) )</td>
<td>( \frac{1}{\sqrt{2\pi}} )</td>
<td>( 1/(4\beta) )</td>
</tr>
<tr>
<td>( N_0 \beta \frac{\sin (-2\pi R \gamma / \beta)}{2\pi \beta} )</td>
<td>( \frac{N_0}{2} \exp \left( \frac{-i \gamma}{2\pi} \right) )</td>
<td>( \frac{1}{\sqrt{2\pi}} )</td>
<td>( 2\pi \beta )</td>
</tr>
</tbody>
</table>

It is important to note that

\[
R(\gamma) = R(-\gamma) \quad |R(\gamma)| < R(0) \quad (1.61)
\]

\[
R(0) = E[x(t)x(t)'] \text{ for a zero mean process}
\]

\[
S(\omega) = S(-\omega)
\]

The cross correlation properties of two wide-sense stationary random processes \( x(t) \) and \( y(t) \) are displayed by the following correlation matrix

\[
K_{xy} = \begin{bmatrix} R_{xx}(\gamma) & R_{xy}(\gamma) \\ R_{yx}(\gamma) & R_{yy}(\gamma) \end{bmatrix}
\]

\[(1.63)\]
\[ R_x(t) = E[x(t)x(t + \tau)] \]
\[ R_y(t) = E[y(t)y(t + \tau)] \]
\[ R_{xy}(t) = E[x(t)y(t + \tau)] \]
\[ R_{yx}(t) = E[y(t)x(t + \tau)] \]  

(1.64)

The probability density function of a Gaussian variable is defined as
\[ p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right] \]

(1.65)

The joint probability density function of two random Gaussian variables with variances \( \sigma_x^2 \) and \( \sigma_y^2 \) and means \( \mu_x \) and \( \mu_y \) is given by
\[ f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp \left[ -\frac{(x - \mu_x)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2} \right] \]

(1.66)

where
\[ p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right] \]

(1.67)

The output \( y(t) \) of a linear time-invariant network, having impulse response \( h(t) \), in response to an ergodic process \( x(t) \) is given by
\[ y(t) = \int h(t - u)x(u) \, du \]

(1.68)

Therefore, the auto-correlation function at the output is
\[ R_y(t) = E[y(t)y(t + \tau)] = \int h(t) \int h(t - u)R_x(u + \tau - u) \, du \, dt \]

(1.69)

where \( R_x(t) \) is the auto-correlation function of \( x(t) \). Thus the spectral density of \( y(t) \) is given by
\[ S_y(f) = \int R_y(t) \exp(-j2\pi ft) \, dt \]

(1.70)

Using (1.90) it can be easily shown that
\[ S_\eta(\omega) = H(\omega) H^*(\omega) S_\xi(\omega) = |H(\omega)|^2 S_\xi(\omega) \] (1.74)

### 1.7 Optimum Linear System

One of the ways of finding an optimum linear system is based upon the criterion of minimizing the mean square error between the input and the output. The error is expressed as

\[ e(t) = f_a(t) - f_d(t) \] (1.72)

where \( f_a(t) \) is the actual output and \( f_d(t) \) is the desired output. Wiener defined the mean square error output as

\[ \bar{e}(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [f_a(t) - f_d(t)]^2 dt \] (1.73)

If \( h(t) \) is the impulse response of the equivalent system between the input and the output, we write (1.73) as

\[ \bar{e}(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} h(s) f_a(t - s) \, ds \, du - f_d(t)^2 \, d\tau \] (1.74)

where \( f_a(t) \) is the input to the system. That is,

\[ f_a(t) \approx h(t) + n(t) \] (1.75)

The condition that gives the minimum value of the mean square error is obtained from (1.74) by applying the techniques of the variational calculus and is given by

\[ R_{nn}(\tau) = \int_{-\infty}^{\infty} h(\omega) R_\eta(\omega - \tau) \, d\omega \] (1.76)

where \( h(\omega) \) stands for the unit impulse response of the optimum system, and \( R_{nn}(\tau) \) is the cross-correlation between the input and the desired output. If the system is causal, i.e.,

\[ h(t) = 0 \text{ for } t < 0 \]

one gets from (1.76)
\[ R_{af}(\tau) = \frac{1}{\pi} \int h(\theta) R_{af} - \theta \, d\theta \quad (1.77) \]

This relationship is known as Wiener-Hopf integral equation. When \( f_a(\tau) \) and \( f_0(\tau) \) are uncorrelated, the solution of this equation leads to the following, provided \( f(\theta) \) and \( h(\theta) \) possess factorizable rational spectral densities.

\[
H(\theta) = \frac{1}{W(0)} \left[ \frac{S_a(\theta)}{W(\theta - \theta)} \right] \quad (1.78)
\]

with

\[
S_a(\theta) + S_0(\theta) = W(\theta) W(-\theta) \quad (1.79)
\]

where \( W(\theta) \) contains all critical frequencies in the left half of the complex frequency plane and \( W(-\theta) \) incorporates all critical frequencies in the right half of the complex frequency plane (s-plane).

\( [\cdots] \) denotes only critical frequencies in the left half of the s-plane. Note that \( H(\theta) \) of (1.78) contains poles in the left half plane. Further, if the noise is white with one sided spectral density \( N_0 \), (1.79) reduces to

\[
H(\theta) = \frac{N_0/2}{N_0/2 + S_0(\theta)} \quad (1.80)
\]

The mean square error is

\[
\overline{e}(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{S_a(\theta)}{S_a(\theta) + S_0(\theta)} + \frac{S_a(\theta)}{W(0)} \right] \overline{S_a(\theta)} \, d\theta \quad (1.81)
\]

[\cdots] denotes the critical frequencies in the right half plane.

Example:

The spectral densities of the signal and the noise at the input to a filter are respectively given by

\[
S_a(\omega) = \frac{1}{\omega^2 + 4}
\]

and

\[
S_0(\omega) = \frac{1}{\omega^2 + 9}
\]

Find the optimum physically realizable system. This is done in the following way:
\[
S(\omega) + S(\omega) = \frac{-1}{\sqrt{2}} \frac{1}{\omega - \frac{1}{2}} \frac{1}{\omega - \frac{1}{2}} \frac{1}{\sqrt{2}} (\sqrt{6.5} + \frac{1}{2}) (\sqrt{6.5} - \frac{1}{2})
\]

Therefore,
\[
W(\omega) = \frac{\sqrt{2}}{(\omega + \frac{1}{2}) (\omega + \frac{1}{2})}
\]

and
\[
W(-\omega) = \frac{\sqrt{2}}{(\omega - \frac{1}{2}) (\omega - \frac{1}{2})}
\]

Hence,
\[
\frac{S(\omega)}{W(-\omega)} = \frac{(\omega - \frac{1}{2})}{\sqrt{2} (\omega + \frac{1}{2}) (\sqrt{6.5} - \frac{1}{2})}
\]

or
\[
\frac{S(\omega)}{W(-\omega)} = \frac{0.777}{\omega + \frac{1}{2}} - \frac{0.07}{\sqrt{6.5}}
\]

Thus the optimum physically realizable filter is (cf. 1.79)
\[
H(\omega) = \frac{0.55}{\sqrt{6.5}} (\omega + \frac{1}{2}) \frac{1}{\sqrt{2}} (\sqrt{6.5} - \frac{1}{2})
\]

Referring to (1.81) it is easily appreciated that the integrand in the ratio of two polynomials in \(\omega\). In this case, the integrand (1.81) can be evaluated by using the following standard results:
\[
L = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} \frac{1}{\omega - \frac{1}{2}} \frac{1}{\omega - \frac{1}{2}} ds
\]

\[
L = \frac{e^{2}d_{p}}{2d_{p}^{4}}
\]

\[
l = \frac{C_{2}d_{p}^{2}}{2d_{p}^{4}}
\]

\[
L = \frac{C_{2}d_{p}^{2} + C_{2}d_{p}^{2} - 2C_{4}d_{p}^{2} + C_{4}d_{p}^{2}}{2d_{p}^{4}d_{p}^{4} - 4d_{p}^{4}}
\]
where
\[ C(n) = C_{n-1}a^{n-1} + C_{n-2}a^{n-2} + \ldots + C_0 \]
\[ d(n) = d_{n-1}a^{n-1} + d_{n-2}a^{n-2} + \ldots + d_0 \]

Further, in the case of white noise of one-sided spectral density, the expression for the minimum mean square error (1.81) reduces to the following
\[ \frac{2}{Z(f)} = \frac{N_0}{2} \int \frac{1 + 2\beta|\omega|/N_0}{2\pi} df \]

1.8 White Noise and Narrowband Noise

Thermal noise or Johnson noise is the unavoidable form of interference that affects the behavior of phase lock systems. Characterization of thermal noise is thus essential and this is provided by thermodynamics and quantum mechanics. Quantum mechanics tells us that the average energy associated with each degree of freedom of a system (say, an electrical network) in thermal equilibrium $T^\circ K$ is given by
\[ E = \frac{\hbar f}{\exp (h/kT) - 1} \]  

where
- $\hbar$ = Planck's constant
  \[ \approx 6.626 \times 10^{-34} \text{ joule-seconds} \]
- $k$ = Boltzmann's constant
  \[ \approx 1.38 \times 10^{-23} \text{ joule/degree Kelvin} \]

Incidentally, in an electrical network consisting of inductances, resistances and capacitances, the number of degrees of freedom is equal to the sum of the number of inductances and capacitances in the network. Inductances in series and capacitances in parallel are considered as a single inductor or capacitor.

Now going back to (1.83), we find that at room temperature ($T = 300^\circ K$) $kT/\hbar$ is $6.25 \times 10^{14}$ Hz and frequencies up to 100 GHz, $E$ can be approximated as
\[ E = kT/\hbar \]
Thus the power spectral density of the thermal noise is \( \frac{4kT}{R} \) and has a value of \( kT/2 \). This is the so-called two-sided power spectral density and the one-sided power spectral density is denoted by \( kT/2 \) (1.83) since this noise is frequency independent, it is called white noise.

The mean square value of the voltage due to thermal noise, measured across a resistance \( R \) in a bandwidth \( B \) is given by

\[
E[U^2(\omega)] = 4kTB
\]

The mean square value of the current that would flow when the resistor is short circuited is

\[
E[I^2(\omega)] = E[U^2(\omega)] = 4kTB
\]  

where

\[
G = 1/R
\]

The thermal noise power available from the resistor \( R \), which is derived by connecting the same resistance \( (R) \) is, therefore, given by

\[
\sigma^2 = E[U^2(\omega)] = 4kTB
\]  

Thermal noise originates from the random motion of conduction band electrons in the resistor; as such the total thermal current produced has a Gaussian amplitude distribution because of the central limit theorem. That is, the probability density distribution of the amplitude is given by

\[
p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}
\]  

where \( kT/2 \) denotes the total thermal power available in the bandwidth \( B \).

White noise is an idealised noise, the spectral density of which is independent of frequency, (cf. 1.84)

\[
S_o(\omega) = \frac{N}{2}
\]

and the autocorrelation function of the white noise is

\[
R_o(r) = \frac{N}{2} \delta(r)
\]
These are shown in Fig. 1.5. Note that the factor $\frac{1}{2}$ indicates that half the power is associated with positive frequency and half with the negative frequency (Fig. 1.5). Consider an ideal bandpass filter with centre frequency $\omega_c$ and bandwidth $2\pi B$ (cf. Fig. 1.6). If a white Gaussian noise of one-sided spectral density $N_0$ is fed to the input of the filter, the output spectral density obtained is shown in Fig. 1.7a. The output of the ideal bandpass filter can be written as

$$n(t) = \sqrt{2N_0} \cos \omega_c t - \sqrt{2N_0} \sin \omega_c t$$

(1.92)

where $n(t)$ and $\eta(t)$ are respectively called the in-phase and quadrature components of the narrow-band noise. Note that a noise is defined as narrowband when its bandwidth is much less than the centre frequency, i.e., $2\pi B \ll \omega_c$. Some properties of $n(t)$ and $\eta(t)$ are summarized below:

1. The spectra of $n(t)$ and $\eta(t)$ are low-pass in nature.
2. If $n(t)$ is Gaussian, then $n(t)$ and $\eta(t)$ are also Gaussian.
3. If $n(t)$ is a zero mean process that $n(t)$ and $\eta(t)$ have zero mean.
4. $n(t)$ and $\eta(t)$ are independent variables.
5. Spectral densities of $n(t)$, $n(t)$ and $\eta(t)$ are given by...
Fig. 1.7. (a) The computed spectral density of white noise when transmitted through an ideal bandpass filter. (b) The spectral density of the quadrature components of the bandlimited noise.

\[ S_n(u) = \begin{cases} \frac{N_0}{2} & \text{if } |u_1 - \pi B| < |u| < (u_0 + \pi B) \\ \text{elsewhere} & \end{cases} \]

\[ S_{\text{oa}}(u) = S_{\text{oa}}(u) = \frac{N_0}{2} \]

i.e., for a symmetrical bandpass filter of Fig. 1.6.

\[ S_{\text{oa}}(u) = S_{\text{oa}}(u) = \frac{N_0}{2} \]

Referring to Fig. 1.7, the auto-correlation functions are given by

\[ R_u(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{\text{oa}}(u) \exp(-juv) \, du \]

\[ = N_0 \frac{\sin(\pi B v)}{\pi B} \cos \omega T \]

(1.96)
26 Phase Lock Theory and Applications

\[ R_a(t) = R_a(0) = \frac{N_1 B}{\pi N_2} \sin \left( \frac{\pi B t}{N_1} \right) \quad (1.97) \]

That is,

\[ \lambda(t) = R_a(0) - i R_0(0) \quad (1.98) \]

The narrowband noise process may be represented in terms of its envelope and phase components in the following way

\[ m(t) = \sqrt{2} \xi(t) \cos (\omega_0 t + \Phi(t)) \quad (1.99) \]

where

\[ r(t) = n_e(t) + n_0(t) \quad (1.100) \]

and

\[ \Phi(t) = \arctan \left[ \frac{\xi(t)}{n_0(t)} \right] \quad (1.101) \]

Now it is known that

\[ p(\xi, \Phi) \, d\xi \, d\Phi = p(r, \Phi) \, dr \, d\Phi \]

Thus it is easily shown (cf. Fig. 1.3) that

\[ f_2(t) \]

\[ f_1(t) \]

Fig. 1.3. Diagrams illustrating calculation of probability density functions of the envelope and phase of narrowband noise.

\[ p(r, \Phi) = \frac{r^2}{\sigma_1 \sigma_2} \exp \left( -r^2/(2 \sigma_1 \sigma_2) \right) \]

and hence
\[ p(r) = \int p(r, \Psi) \, d\Psi = \frac{\alpha^n}{\pi^{\frac{n}{2}}} \exp \left( - \frac{r^2}{2\alpha^2} \right) \quad r > 0 \quad (1.103) \]

and

\[ p(\Psi) = \int p(r, \Psi) \, dr = \frac{1}{2\pi}, \quad 0 \leq \Psi < 2\pi \]

\[ = 0 \quad \text{elsewhere} \]

Putting

\[ \lambda = \frac{R_n(\gamma)}{\alpha^2} = \frac{R_n(\gamma)}{\alpha^2} \]

it is shown that the correlation functions of \( r(t) \) and \( \Psi(t) \) are given by

\[ R_n(r) = \frac{n^2}{2} \left\{ 1 + \left[ 1 - R_n(0) + \sum \frac{1}{2\pi} R_n(\gamma) \right] \frac{r^2}{2\pi} \right\} \]

\[ + \frac{1}{2\pi} \left( \sum \frac{1}{2\pi} R_n(\gamma) \right) r^2 \quad (1.104) \]

and

\[ R_n(\gamma) = \frac{n^2}{2} \pi n(t) + \frac{n}{2} \pi n^2(t) + \sum \left( \frac{1}{2\pi} R_n(\gamma) \right) \pi n^2(t) + \ldots \quad (1.105) \]

where \( n^2 \) is the variance of \( n(t) \) or \( n_0^2(t) \), i.e.,

\[ \sigma^2 = \lambda_0(0) = R_n(0) \]

The probability density function of (1.102) is known as the Rayleigh distribution function (cf. Fig. 1.9). Similarly, the probability density function of the sine wave plus a narrow-band noise can be easily obtained by writing

\[ u(t) = \sqrt{2\alpha^2} \cos \omega t + n_0(t) \cos \omega t - n_0(t) \sin \omega t \]

\[ = \sqrt{2\alpha^2} \cos (\omega t + \Psi(t)) \quad (1.106) \]

where

\[ r(t) = (A_0 + n_0(t))^2 + n_0^2(t) \]

and

\[ \Psi(t) = \text{atan} \left( \frac{n_0(t)}{A_0 + n_0(t)} \right) \quad (1.107) \]

The joint density function of \( r \) and \( \Psi \) can be shown to be given by
\[ p(r, \Psi) = \frac{r}{2\pi} \exp \left[ - \left( r^2 + A^2 - 2rA \cos \Psi \right) / 2a^2 \right] \]

Thus the pdf of \( r \) becomes

\[ p(r) = \frac{r}{2a} I_0 \left( \frac{rA}{2a} \right) \exp \left( - \frac{r^2 + A^2}{2a^2} \right) \] (1.108)

and

\[ p(\Psi) = \frac{\exp \left( - A^2 / 2a^2 \right)}{2a} \frac{A \cos \Psi}{\sqrt{2\pi}} \left[ 1 + \text{erf} \left( \frac{A \cos \Psi}{\sqrt{2a^2}} \right) \right]. \] (1.109)

where

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp \left( - z^2 \right) dz \] (1.110)

and \( I_0(x) \) is the modified Bessel function of order 0 and argument \( x \).

The probability density function of (1.108) is known as the Rician distribution function, which is shown in Fig. 1.10. Note that when no signal is present, the Rician distribution becomes the Rayleigh distribution, whereas the signal power is very large compared to \( \sigma^2 \), the Rician distribution approaches the Gaussian distribution with a mean \( A \).
1.9 Noise Bandwidth

Let white Gaussian noise of one-sided power spectral density $N_0$ be fed to a linear time-invariant network with transfer function $H(j\omega)$, as shown by the solid lines in Fig. 1.11(a) and Fig. 1.11(b). The power spectral density at the output of the system is given by

$$ S(\omega) = \frac{N_0}{2} |H(j\omega)|^2 \quad (1.11) $$

and is shown in Fig. 1.12(a) and Fig. 1.12(b). Thus the noise power at the output of the filter is given by

$$ \sigma^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \, d\omega \quad (1.12) $$

since $|H(j\omega)|^2$ is an even function of $\omega$, (1.79) can be written as

$$ \sigma^2 = \frac{N_0}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2 \, d\omega \quad (1.13) $$

Now we define a quantity called the loop bandwidth $2\omega_c$ in the following way.
\[ |H(f)|^2 \]

(a) Low Pass

\[ |H(f)|^2 \]

(b) Band Pass

Fig. 1.11 Illustrating equivalent noise bandwidths in (a) the lowpass filter and (b) the bandpass filter.

\[ B_L = 2B_r = \frac{1}{2\pi} \int_0^\infty |H(j\omega)|^2 d\omega \]  

(1.114)

Comparing (1.103) and (1.105) we find

\[ \sigma^2 = \frac{N}{2} B_L H_r^2 = N_r R_H H_r^2 \]  

(1.115)

Thus \( B_L \) equals the bandwidth of an ideal rectangular filter, that would produce the same noise power at the output as the real system would do. \( B_r \) is obviously the single-sided bandwidth. \( B_r \) is often called the noise bandwidth of the system.
Fig. 1.12. The output equivalent spectral density of white noise passed through the lowpass and bandpass filters.

REFERENCES

32 Phase Lock Theories and Applications


NEARLY SINUSOIDAL OSCILLATIONS

2.1 Introduction

A nearly sinusoidal oscillator may be looked upon as a device that generates an almost sinusoidal function of time. Thus it is a regenerative feedback device that derives its required input from its own output either by way of internal feedback mechanism or through an external feedback network connecting its output to the input port. Depending upon these modes of feedback process one may classify oscillators under two broad heads, viz., Negative Resistance Oscillators and Feedback Oscillators. Thus, whatever is the type of nearly sinusoidal oscillators, it is always a positive feedback device incorporating a frequency selective network to pick up the desired frequency of oscillation and a limiter type non-linear element to check the growth of oscillation to a suitable level. The process of limiting may be instantaneous or non-instantaneous. Examples for instantaneous and non-instantaneous limiting processes may be provided by referring to class-A and class-C operations of oscillators, respectively.

2.2 Sinusoidal Oscillations in Linear Systems

Consider a mass $m$ attached to the end of a spring, as shown in Fig. 2.1a, whose extension is strictly linearly proportional to the force applied by weight of the mass $m$. How if one displaces the weight and releases it, as shown in Fig. 2.1a, it will execute motion determined by Newton's law, that is, acceleration = force, being proportional to the displacement $x$, divided by the mass of the weight.
Fig. 21. Oscillations in: (a) a massless spring, (b) an L = C circuit, and (c) an ideal pendulum.

Hence one writes,

\[ \frac{d^2x}{dt^2} = -kx/m. \]  \hspace{1cm} (2.1)

Note that \( k \) is a constant of proportionality that depends on the inherent property of the spring. Thus one writes

\[ \frac{d^2x}{dt^2} + \omega_0^2 x = 0, \]  \hspace{1cm} (2.2)

where

\[ \omega_0^2 = k/m. \]

Now consider the case of discharging a capacitor \( C \) having an initial charge \( Q_0 \) through a lossless inductance \( L \). This is illustrated in Fig. 21b. In this case, by applying Kirchhoff's law, one finds that

\[ \frac{\dot{q}}{C} + L \frac{d^2q}{dt^2} = 0, \]

or

\[ \frac{d^2q}{dt^2} + \omega_0^2 q = 0, \]  \hspace{1cm} (2.3)

where \( \omega_0 = \frac{1}{\sqrt{LC}} \) and \( q \) is the instantaneous charge residing on the plates of the capacitor. By virtue of the functional similarity between (2.2) and (2.3), we say that the charge \( q \) is the analog of the displacement \( x \), the inductance \( L \) is the analog of the mass \( m \) and the
stiffness $k$ of the spring is the analog of the inverse of the capacitance $C$. Anyway, since the equations (2.3) and (2.4) are identical in form we rewrite either of them as

$$\frac{du}{dt} + au = 0,$$

where $u$ signifies either the displacement of the spring or the charge on the condenser plates and $a$ denotes either the underride of $\frac{1}{k}$ or of $\frac{1}{C}$.

Now the general solution of the differential equation (2.4) is known to be given by

$$u = A \cos at + B \sin at,$$

where $A$ and $B$ are disposable constants to be determined from the initial conditions of the problem. For example, if the weight is initially taken to a displacement of $X$ and then released, then one would get

$$x = X \cos at,$$

or for the case of discharge of the condenser from an initial charge of $Q$, one finds

$$q = Q \cos at.$$

From the nature of the solution, one finds that the period $2\pi/a$ does not depend on the amplitude of oscillations ($X$ or $Q$). Moreover, it is seen that all the amplitudes of oscillations are possible, depending upon the initial condition of the problem. However, in such case the frequency of oscillation is the same. That is, we say that each of the periodic solutions is isolated. This is the characteristic property of linear differential equations and hence of linear systems.

Let us now consider the motion of a simple pendulum in a frictionless medium. The equation of motion is known to be of the form

$$\frac{d\theta}{d\tau} + u^2 \sin \theta = 0,$$

where

$$\theta = \text{the deflection of the pendulum from the vertical plane passing through the point of suspension.}$$
\[ a^2 = \frac{1}{l} \]

\[ m = \text{mass of the pendulum} \]

\[ g = \text{acceleration due to gravity} \]

\[ l = \text{length of the pendulum} \]

Multiplying both sides of (2.7) by \( 2(d^2\theta)/dt^2 \) one gets

\[
\frac{d}{dt} \left( \frac{d\theta}{dt} \right)^2 = 2a^2 \sin \theta \frac{d\theta}{dt} \tag{2.8}
\]

Assuming \( \alpha \) to be the maximum angle of displacement when \( d\theta/dt = 0 \), we can integrate (2.8) to write

\[
\frac{d\theta}{dt} = 2a^2 (\cos \theta \_ \cos \alpha) \tag{2.9}
\]

i.e.,

\[
\frac{d\theta}{dt} = 2a \sqrt{\cos \theta - \cos \alpha} \tag{2.9a}
\]

Therefore, the period of oscillation is given by

\[
T = \frac{2a}{\sqrt{\cos \theta - \cos \alpha}} \tag{2.10}
\]

using the identity \( \cos \alpha = 1 - 2 \sin^2(\alpha/2) \) and putting

\[ k = \sin \alpha/2 \]

\[ \sin \theta/2 = k \sin \varphi \]

we get

\[
T = \frac{2a}{k} \sqrt{\frac{d\theta}{dt}} \tag{2.11}
\]

The right hand side of (2.11) is an elliptic integral of the first kind as tabulated in the Tables of elliptic integrals.

For small values of \( k \), a series solution of (2.11) is obtained by expanding the radical, and we get

\[
T = \frac{2a}{k} \left[ 1 + \frac{k^4}{16} + \ldots \right] \tag{2.12}
\]

The equation of motion of the pendulum (2.7) is a non-linear differential equation. We observe that the time period depends on
the amplitude of oscillation, i.e., on the maximum displacement. In passing we mention that only a few non-linear equations have known solutions.

Thus far we have looked into the behavior of an oscillating system from the analytical point of view. We will now explore the oscillating system from the geometric point of view, developed by the French scientist Poincare and the Russian scientist Liapunov. Here one will see that we do not have to solve the differential equation, instead all the properties of the system will follow automatically from the geometric constructions.

2.3 Phase Plane or State Space Approach

Let us now introduce a new symbol \( r = \frac{dy}{dt} \) and rewrite equation (2.4) as two first order equations, viz.,

\[
\frac{dx}{dt} = v
\]

and

\[
\frac{dv}{dt} = - w^2 x.
\]

(2.12)

(2.14)

Note that \( u \) and \( v \) signify the state of the system, for example, (i) in the case of the oscillating mass, \( u = x \) and \( v = \frac{dx}{dt} \) denote respectively the position and the velocity of the weight and (ii) in the case of the discharging condenser, \( u = q \) and \( v = \frac{dy}{dt} \) signify respectively the charge and the current of the system. These are the minimum data required to define the behavior of the system completely. Thus in the \( u \rightarrow v \) plane, each point represents the state of the system. This plane is called either a phase plane or a state space [1, 2]. The differential equations can be pictured upon as a flow in the plane. Referring to (2.13) and (2.14) one finds that

\[
\frac{du}{dv} = - \frac{v}{w^2 u},
\]

(2.15)

which on integration can be written as

\[
\omega^2 u^2 + v^2 = \text{Const} = \Delta_0^2 \text{ (say)},
\]

(2.16)

\[
u^2 + \left(\frac{v}{w^2 u}\right)^2 = \text{Const} = \Delta^2,
\]

(2.17)
This represents a number of concentric circles of different radii \( R_n \). The values of the radii will be determined by the initial conditions of the problem (cf. Fig. 2.2a). The time to trace out an angle \( \theta \) i.e., a portion of the arc \( \theta R \) of the circle, is (cf. 2.13).

\[ t_{ab} = \int \frac{du}{v}. \quad (2.18) \]

Hence, substituting \( v \) from (2.16) in (2.18) one gets

\[ t_{ab} = 0/ua. \quad (2.19) \]

Therefore, the time period, i.e., the time for tracking out a complete circle is, \( (\theta = 2\pi) \)

\[ T = 2\pi/ua. \quad (2.20) \]

This indicates that the time period is independent of the trace considered, i.e., independent of the initial condition. That is, the oscillations are isoperiodic.

We consider the equation of motion of the pendulum and rewrite them in the following form.

On putting

\[ \frac{d\theta}{dt} = f \]

and inserting in (2.7) we find

\[ \frac{df}{dt} = -\omega^2 \sin \theta \]

Hence from (2.21) and (2.22), we write

\[ \frac{df}{d\theta} = -\frac{\omega^2 \sin \theta}{\theta} \quad (2.23) \]

This is plotted in \( \theta = \theta \) plane as shown in Fig. 2.2b, which gives the phase plane portrait. Equation (2.23) may be integrated to yield

\[ \frac{\theta^2}{2} = \omega^2 \cos \theta + \text{const} \]

Assume the following boundary condition, i.e., at time \( t = 0 \)

\[ \theta = 0, \quad \phi = \Omega_0 \]

Therefore, (2.24) reduces to
\[ \theta = \Omega_0 \left( 1 - \left( \frac{\omega}{\Omega_0} \right)^2 \sin^2 (\theta/2) \right)^{1/2} \]  

(2.25)

Thus depending upon whether \(4\omega^2/\Omega_0^2\) (i.e., twice the maximum potential energy divided by the initial kinetic energy) is greater, equal or less than unity, the pendulum executes three types of motion, namely,

(i) When \(4\omega^2/\Omega_0^2 > 1\), the pendulum oscillates with a maximum angle of deflection \(\theta_0 = 2 \arcsin (\Omega_0/2\omega)\).

(ii) When \(4\omega^2/\Omega_0^2 < 1\), the pendulum rotates about its point of suspension because of its high initial velocity \(\Omega_0\). In this case the angular displacement increases or decreases depending upon clockwise and counterclockwise rotation. However, the angular velocity fluctuates about a mean value. We note that when \(4\omega^2/\Omega_0^2 < 1\), \(\theta\) never becomes zero. The curves are open as shown in Fig. 2.15 and the motion of a point \((\theta, \dot{\theta})\) takes place from left to right in the upper half plane, whereas in the lower half plane it is from right to left. We again find that when \(4\omega^2/\Omega_0^2 > 1\), the curves are closed in the phase plane. The transition boundary, that separates the closed curves from the open ones is obtained by taking \(4\omega^2 = \Omega_0^2\). That is, the transition boundary is defined by (cf. 2.17):

\[ \dot{\theta}^2 = \Omega_0^2 \cos^2 (\theta/2) \]  

(2.26)

This trajectory is sometimes referred to as the separatrix.

The type of equation which is amenable to phase plane analysis is of the form

\[ f(x, \dot{x}, \ddot{x}) = 0 \]  

(2.27)

As a result of the form of the equation (2.25), the phase plane method has the following restrictions:

(1) Since two dimensions are used to sketch the phase plane portrait, second order differential equations can be studied by this method. The analysis of higher order systems requires multidimensional space.

(2) Only signal dependent non-linear differential equations can be studied by this method.

(3) Transient responses of a system subject to initial conditions, but otherwise unexcited, can be studied with the help of this system.

Substituting

\[ x = y, \quad \dot{x} = \frac{dy}{dt}, \quad \frac{\partial y}{\partial x} = -\frac{\partial y}{\partial x} \]  

(2.27)
the Eqn. (2.27) of a second order autonomous dynamical system can be written as

\[ f(x, y, \frac{dy}{dx}) = 0 \]  

(2.28)

Equation (2.28), when solved for \( \frac{dy}{dx} \), gives in general

\[ \frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} \]  

(2.29)

which can also be written as

\[ \frac{dy}{dx} = P(x, \bar{y}) \]  

(2.30)

\[ \frac{d\bar{y}}{dt} = Q(x, \bar{y}) \]  

(2.31)

In passing it is to be noted that the singular points of (2.29), that are equilibrium states of the system, are given by the points of intersection of the curves

\[ P(x, \bar{y}) = 0 \]  

(2.32)

and

\[ Q(x, \bar{y}) = 0 \]  

(2.33)

in the phase plane \((x, \bar{y})\) plane, since at these points \( dy/dx \) of (2.29) is undefined. The points of intersections are also called singularities or critical points. Note that if the initial values of \( x \) and \( \bar{y} \) correspond to those of the critical points, the system will remain the same since at these points \( x = 0 \) and \( \bar{y} = 0 \). Now let us see what will happen when the point of observation \((x, \bar{y})\) is the plane lies close to the points of equilibrium \((x_0, \bar{y}_0)\). There are three possibilities, viz.,

1. The distance between \((x, \bar{y})\) and \((x_0, \bar{y}_0)\) may increase with time. The point \((x_0, \bar{y}_0)\) is called unstable.

2. The point \((x, \bar{y})\) converges to \((x_0, \bar{y}_0)\). The point \((x_0, \bar{y}_0)\) is called asymptotically stable.

3. The distance between \((x, \bar{y})\) and \((x_0, \bar{y}_0)\) varies with time but remains between an upper and a lower bound. The point \((x_0, \bar{y}_0)\) is called neutrally stable.

The nature of stability in the proximiy of the equilibrium point can be studied by linearizing (2.30) and (2.31). Let

\[ x = x_0 + \xi \]
where $\xi$ and $\eta$ are arbitrarily small quantities. Therefore, substituting this in (2.30) and (2.31) and noting that

$$\frac{dx_1}{dt} = P(x_1, x_2),$$
$$\frac{dx_2}{dt} = Q(x_1, x_2)$$

and $x_1$ and $x_2$ are arbitrarily small parameters, one finds after discarding higher order terms in the Taylor's series expansions of $P(x_1, x_2)$ and $Q(x_1, x_2)$ around the point $(x_0, \dot{x}_0)$,

$$\frac{dx_1}{dt} = a_1 x_1 + b_1$$
$$\frac{dx_2}{dt} = c_1 x_2 + d_1$$

where

$$a = \frac{dp}{dx} \bigg|_{x=x_0}$$
$$b = \frac{dp}{dx} \bigg|_{x=x_0}$$
$$c = \frac{dQ}{dx} \bigg|_{x=x_0}$$
$$d = \frac{dQ}{dx} \bigg|_{x=x_0}$$

and

$$\Phi = (a + c) x + (ad - bc) = 0$$

The roots of (2.37), which are given by

$$x_1, x_2 = \frac{\alpha + \beta \pm \sqrt{\alpha^2 + \beta^2}}{2}$$

(2.38)

determine the nature of the critical point. A knowledge of the nature of the roots $x_1$ and $x_2$ allows to sketch the trajectories in the neighbourhood of the equilibrium point. Looking at (2.38) it is easily seen that there are six possible cases depending on the sign of $(a - \alpha)$ and $(ad - bc)$, and their relation. This is illustrated in Fig. 2.2c and
Fig. 2.2. (a) The phase plane trajectories of the oscillating mass and charge, (b) the oscillating pendulum, (c) explaining significance of the root of the characteristic equation.

The corresponding phase plane trajectories [1, 2, 7, 10, 11] in the neighborhood of the critical points are shown in Fig. 2.3. Depending upon the nature of roots, the trajectories assume different shapes, and the various configurations have different nomenclature as shown in Fig. 1.2 c and Fig. 2.3. As the trajectories which either converge or diverge away from the equilibrium point, there is an interesting set of trajectories, which close in themselves and correspond to periodic oscillatory motion. These trajectories are called periodic trajectories. These are illustrated in Fig. 2.4. The motion of stability is applicable here, but with the following points of difference arising out of the division of the phase plane by two parts by periodic trajectories: inside and outside. Thus one can have stability both
from inside and outside (Fig. 2.4a), or one can have stability from outside but instability from the inside (cf. Fig. 2.4b) and vice-versa (cf. 2.4c). Periodic trajectories which are asymptotically stable from both inside and outside are called limit cycles. Those which are un-
stable from both sides are called antilimit cycles. We will discuss this in the sections to follow.

2.4 Nearly Sinusoidal Oscillations

Consider the two terminal negative resistance oscillators of Fig. 2.5a and Fig. 2.6a. Fig. 2.5a represents a conventional dynatron oscillator, whereas Fig. 2.6a depicts a microwave solid state oscillator, using a Gunn or an IMPATT diode. Obviously, these are two terminal oscillators. The negative resistance reflected by the tetrode is a voltage-controlled one in the sense that it depends on the voltage across the plate and the cathode. On the other hand, the negative resistance presented by the Gunn diode is a current-controlled one. These are shown in Fig. 2.5c and Fig. 2.6e respectively. Referring to the equivalent circuit of the two oscillators as shown in Fig. 2.5b one can easily show that (see Appendix A)
Fig. 2.5. (a) The dynatron oscillator, (b) its equivalent circuit, and (c) its voltage dependent resistance characteristic.

Fig. 2.6. (a) Schematic representation of a Gunn or IMPATT oscillator, (b) its equivalent circuit, and (c) its current dependent resistance characteristic.
\[
\frac{d^2 v}{dt^2} + \frac{1}{C} \frac{d}{dt} \left( \frac{RC}{L} v - F(v) \right) + a_v^2 [1 - RF(v)] = 0,
\]  
where 
\[
a_v^2 = \frac{1}{LC}
\]  
and \( F(v) \) represents the nonlinear function representing the relation between the plate current and plate voltage of the device. Referring to Fig. 2.6b one easily finds (Appendix-A) that 
\[
\frac{d^2 q}{dt^2} + Q \left[ \frac{R_k + R - R_v}{R_k} \frac{1}{R_k} \frac{d}{dt} + \frac{1}{R_k} \frac{d}{dt} \left( \frac{d}{dt} \right)^2 \right] \frac{dq}{dt} + a_v^2 [(1 + C_{\text{eq}}) + C_{\text{eq}} v] q = 0.
\]  
Note that \( q \) represents the instantaneous charge circulating through the equivalent circuit of the Gunn diode oscillator. Referring to (2.39) and (2.41) one finds that both these equations are nonlinear differential equations. For the moment, we do not attempt to find the nature of the solutions. Instead, we are referring to the feedback type of oscillators of Fig. 2.7 and Fig. 2.8. Referring to Fig. 2.7, it is not difficult to show that the triode oscillator equation is (see Appendix-B).

Fig. 2.7. The plate tuned vacuum tube oscillator.

\[
\frac{d^2 x}{dt^2} + \frac{1}{C} \left( \frac{d}{dt} - \frac{dF(v)}{dt} \right) + a^2 v = 0.
\]
where
\[ v_0^2 = \frac{1}{L_C}, \quad v_0^2 = v_0^2 \left( 1 + \frac{R}{R_p} \right) \]
and
\[ V_p = V_{pp} - \frac{L_nR}{\mu} + V_{DD} \]
\( \mu \) is the amplification factor of the tube and \( R_p \) is the plate resistance of the same. The significance of other parameters are stated in Appendix-B.

Let us define the variables
\[ v_0 = \frac{\mu M - L}{\mu M} v_p, \quad N(v_p) = \eta f(v_p + v_0) \quad \text{and} \quad \eta = \frac{\mu M - L}{\mu C R} \]
and using (2.42) one has
\[ \frac{d^2v_p}{dt^2} + \frac{\mu M}{\mu M} \frac{d}{dt} v_0 = N(v_p) \quad \text{and} \quad \frac{d^2v_p}{dt^2} + \frac{\mu M}{\mu M} \frac{d}{dt} v_0 = 0. \]  
(2.43)

In practice \( r_p \gg R \) such that \( v_0 \approx u_0 \). Thus one may rewrite (2.43) as
\[ \frac{d^2v_p}{dt^2} + \frac{u_0}{Q} \frac{d}{dt} v_0 = N(v_p) \quad \text{and} \quad \frac{d^2v_p}{dt^2} + \frac{u_0}{Q} \frac{d}{dt} v_0 = 0. \]  
(2.44)

Similarly, referring to Appendix-B one gets the transistor oscillator equation as
\[ \frac{d^2v_p}{dt^2} + \frac{u_0}{Q} \frac{d}{dt} v_0 = N(v_p) \quad \text{and} \quad \frac{d^2v_p}{dt^2} + \frac{u_0}{Q} \frac{d}{dt} v_0 = 0. \]  
(2.45)
2.5 Solutions of the Oscillator Equations

We have already noted that the equations governing the behaviour of the two terminal or four terminal oscillators are nonlinear differential equations. In the following we will discuss the nature of the solutions \([1, 2, 11]\). However, we will not discuss the behaviour of the Gunn diode oscillator. The interested reader may refer to the references \([3, 4, 5]\) listed at the end.

Referring to the equations (2.39), (2.42) and (2.45), we find that the functional forms of these equations are the same. Therefore, we rewrite them in the following common form

\[
\frac{d^2x}{dt^2} + \omega_0^2 \frac{d}{dt} (x - \chi(x_0 + x)) + \omega_4^2 x = 0. \tag{2.46}
\]

\(x_0\) indicates the location of the operating point.

A typical nature of the nonlinear function \(f(x)\) is shown in Fig. 2.9.

Now expanding the function \(f(x)\) around \(x_0\) by Taylor's series, one finds that

\[
f(x) = f(x_0) + \frac{df}{dx}(x_0)(x - x_0) + \frac{d^2f}{dx^2}(x_0) + \ldots.
\]

Equation (2.46), thus, can be written as

\[
\frac{d^2x}{dt^2} + \omega_0^2 \left[ \frac{df}{dx}(x_0) \right] + \omega_4^2 x = 0, \tag{2.47}
\]

where

\[
C_s = \nu f(x_0). \tag{2.48}
\]
The differential equation (2.47) is a nonlinear one with nth order polynomial in the variable (x). It is impossible to find a general solution of the equation. In view of this, we will take an example of a practical oscillator, and we will consider three different cases depending upon the operation of the oscillator around the point either (A), (B) or (C) of Fig. 2.9.

2.5.1 Oscillator Operating Around the Quiescent Point (A) (cf. Fig 2.9)

In such a case, the nth order polynomial representation of the nonlinearity can be approximated by

\[ yC_nx^n = C_0 + C_1x - C_2x^2. \]  

(2.51)

where

\[ C_0 = \eta^2 \left( \frac{B}{B_0} \right). \]

(2.52)

\[ C_1 = \eta \left( \frac{B}{B_0} \right) - 1. \]

(2.53)

\[ C_2 = \eta \left( \frac{B}{B_0} \right)^2. \]

(2.54)

Therefore, (2.47) can be written as

\[ \frac{d^2x}{dt^2} - \frac{n}{2} \frac{dx}{dt} \left[ C_0x - C_2x^2 \right] + n^2x = 0. \]

(2.55)

Putting

\[ \mu = \frac{C_0x}{C_1x^2} \quad \text{and} \quad \epsilon = \frac{C_2}{C_1}, \]

one can rewrite (2.55) as

\[ \frac{d^2x}{dt^2} - \alpha \frac{dx}{dt} + \epsilon n^2x = 0. \]

(2.56)

Equation (2.56) is known to be the van der Pol's differential equation for the oscillator [14, 15].

Observe that integration of the above equation (cf. (2.50)) is not possible because of its nonlinear nature. However, useful information
can be had for small values of $\epsilon$. For example, if $\epsilon$ is zero, (2.56) reduces to

$$\frac{d^2u}{dt^2} + \omega^2 u = 0,$$  \hspace{1cm} (2.57)

the solution of which is indicated in sections 2.2 and 3.3 (refer to 2.5) or (2.5a). Let us now look into the nature of the van der Pol's equation for very small values of $u$, in which case one may rewrite (2.56) as

$$\frac{d^2u}{dt^2} - \epsilon \frac{du}{dt} + \omega^2 u = 0,$$  \hspace{1cm} (2.58)

the solution of which is known to be given by

$$u = A \exp \left[\frac{\epsilon}{2t} + \left(\frac{\epsilon^2}{4} - 1\right)^{\frac{1}{2}}\right] + B \exp \left[\frac{\epsilon}{2t} - \left(\frac{\epsilon^2}{4} - 1\right)^{\frac{1}{2}}\right].$$  \hspace{1cm} (2.59)

This indicates that $u$ will start growing from very small initial value. The rate of growth with time obviously depends on the value of $\epsilon$. Naturally, the larger the value of $\epsilon$, the faster will be the growth of $u$. But as soon as $u$ becomes greater than unity, the coefficient of $du/dt$ becomes positive (cf. 2.56). This indicates that the amplitude of $u$ will then start decaying. For larger values of $\epsilon$, the decay will be faster. However, when it again becomes less than unity it will again start growing. In this way it is expected that the nonlinear differential equation will have a steady state solution. Here one would observe that the nonlinearity of the differential equation curbs the growth of oscillation.

Since analytical solution of the van der Pol's equationET for any value of $\epsilon$ is not possible, computer solutions of the same are shown in Figs. 2.10a, Fig. 2.10b and Fig. 2.10c for $\epsilon = -1.1$ and 10 respectively.

2.6 Phase Plane Plots of the van der Pol's Equation

Let us rewrite the van der Pol's equation (2.56) in a more convenient form by putting

$$\tau = ut$$  \hspace{1cm} and  $$\gamma = \epsilon u,$$

thus we get

$$\frac{d\tau}{d\tau} - \gamma(1 - \omega^2) \frac{d\tau}{d\tau} + \tau = 0$$  \hspace{1cm} (2.60)
Equation (2.60) can be written as
\[ \frac{dv}{dt} = \gamma(1 - u^2) v - u \]  
(2.61)

where
\[ v = \frac{du}{dv} \]  
(2.62)

Thus we get
\[ \frac{dv}{dt} = \gamma(1 - u^2) v - u \]  
(2.63)

The plot of this equation is shown in Fig. 2.11 for small values of \( \gamma \), say 0.1. From the phase trajectories one finds that, when \( u \) and \( v \) are small, the phase portrait spirals out as indicated, by the dotted curve and finally merges with the circle \( C \). Similarly, for large values of \( u \) and \( v \) the phase trajectory spirals in following the dashed line of Fig. 2.11 and ultimately unites with the circle \( C \). The isolated, closed curve, i.e., here the circle \( C \), to which the trajectories merge, emanating either from inside or from outside of the closed curve, is called a limit cycle. The limit cycle operation of a system obviously indicates the oscillatory motion of a system. In particular, the circular path of the limit cycle indicates a sinusoidal oscillation.

If one takes a larger value of \( \gamma \), the phase plane trajectories will be
as shown in Fig. 2.12. Here also the limit cycle operation is seen, but the limit cycle is not a circle. This indicates that the motion will be oscillatory but not a sinusoidal one. Thus, referring to either Fig. 2.11 or Fig. 2.12, one finds that there is only one closed solution curve and all other solutions except \( u = 0 \) and \( \omega = 0 \) approach the limit cycle as the time \( t \) approaches infinity, i.e., in the steady state.
The existence of a limit cycle may be proved in the following way. Consider Fig. 2.13. If a limit cycle exists within the region bounded by circles $C_1$ and $C_2$, it means that all the paths outside the circle $C_2$ must converge towards the limit cycle and inside the circle $C_1$, all the paths are diverging. This indicates that the radial distance of a point $P_n$ moving along a path outside the circle $C_2$ will be decreasing. Whereas the radial distance of a point $P_1$ lying on a path inside the circle $C_1$ will be increasing. Based on this concept, we demonstrate the required convergence and divergence properties, taking the example of the van der Pol's oscillator.

The radial distance of a point in the $xy$ plane is given by

$$\frac{dr}{dt} = \frac{d}{dt} (x^2 + y^2) = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

(2.64)

Comparing this with (2.61) and (2.63), it is seen that,

$$\frac{dr}{dt} = 2\gamma x^2 (1 - u^2)$$

(2.65)

Thus for positive values of $\gamma$, the radius vector $r$ will be shrinking since $\frac{dr}{dt}$ is negative for $u > 1$. Similarly, when $\gamma$ is very small, the radius vector will be increasing. This indicates that at least one
limit cycle must exist. This method of establishing the existence of a limit cycle becomes cumbersome in many cases. There are several theorems for tackling this problem [1, 2]. A full exposition of the various methods is outside the scope of this book. Pointare discovered certain properties of limit cycles which were later supplemented by the works of Baudischon, Andronov and others. Their findings are summarised in [7].

Thus far we have illustrated the method of solving the van der Pol’s equation, a non-linear differential equation, through phase plane method. In the following section we will describe a method of solving the van der Pol’s equation when \( \epsilon \) is small, i.e., when the system operates under nearly sinusoidal oscillation.

2.7 Asymptotic Techniques

Observe that quantitative integration of the nonlinear differential equation of the form

\[
\frac{dx}{dt} - \epsilon \left( \sum C_i x^i \right) + \omega_0^2 x = 0
\]  
\tag{2.66}

or

\[
\frac{dx}{dt} - \epsilon \int \left( \frac{dx}{dt} \right)^2 + \omega_0^2 x = 0
\]  
\tag{2.67}

is not possible. Moreover, the phase plane method of solution is tedious and time consuming. As such analytical methods [6] for solving (2.66) or (2.67) for small values of \( \epsilon \) (i.e., weakly nonlinear systems) have been proposed. In the following we will describe methods for solving such equations.

When \( \epsilon = 0 \), looking at (2.66) or (2.67) one finds that the solution is of the form

\[
x = A_0 \cos (\omega_0 t + \Psi_0)
\]  
\tag{2.68}

where \( A_0 \) is the amplitude of oscillation at the radius frequency \( \omega_0 \) and \( \Psi_0 \) is the epoch or the phase of the oscillator. Now it seems reasonable to assume that for small values of \( \epsilon \) the solution of (2.66) and (2.67) may be assumed to be of the form:

\[
x = A_0 \epsilon \cos (\omega_0 t + \Psi(\epsilon))
\]  
\tag{2.69}

Before we really go into solving (2.66) or (2.67), let us remember that the nonlinearity of the system checks the growth of oscillation.
As such, the sinusoidal oscillation generated by a nonlinear system cannot be a pure one. Instead there has to be a certain amount of distortion, however, small it may be, associated with the generated sinusoidal oscillations. If this is small, the distortion is small. In such a situation, we speak of what is known as nearly sinusoidal oscillation. Therefore, the assumed solution (3.59) will give a nearly correct picture of oscillations but not the accurate one, at least when it has reached a fairly large amplitude. But during its build-up it will give an almost accurate picture of the phenomenon of oscillation in a weakly coupled oscillator. Now from (3.69) one has

\[
\frac{dx}{dt} = \frac{dA}{dt} \cos (\omega_0 t + \Theta) - \omega_0 A \sin (\omega_0 t + \Theta)
\]

(2.70)

\[
\frac{d^2x}{dt^2} = -2\omega_0 \frac{dA}{dt} \sin (\omega_0 t + \Theta)
\]

\[- A \left( \omega_0^2 + 2\omega_0 \frac{d^2A}{dt^2} \right) \cos (\omega_0 t + \Theta)
\]

(2.71)

and

\[
\frac{d}{dt} \left[ (2\omega_0^2 A^2) \right] = \xi C\omega A^{\nu=1} \cos (\omega_0 t + \Theta) \frac{dA}{dt}
\]

\[- \xi C\omega A^{\nu=1} \cos (\omega_0 t + \Theta) \sin (\omega_0 t + \Theta) \left[ \omega_0 + \frac{d^2A}{dt^2} \right]
\]

(2.72)

In this derivation \(A(t)\) and \(\Theta(t)\) have been assumed to be slowly varying functions of time. That is

\[
\frac{1}{A} \frac{dA}{dt} \leq \omega_0 \frac{d\Theta}{dt} \leq \omega_0
\]

Combining (2.72), (2.71) and (2.66) and using the method of harmonic balance [9], one gets

\[
\frac{dA}{dt} = \frac{\epsilon}{2\omega_0} \xi C \omega A^{\nu=1} \left[ \cos (\omega_0 t + \Theta) \cos (\omega_1 t + \Theta) \right] dt
\]

(2.73)

and

\[
\frac{d^2A}{dt^2} = -\frac{\epsilon}{2\omega_0} \xi C \omega A^{\nu=1} \frac{dA}{dt} \left[ \omega_0 \cos (\omega_1 t + \Theta) \cos (\omega_0 t + \Theta) \right] dt
\]

(2.74)

The above equation for the amplitude and phase variations can also
be derived following the method of Krylov and Bogoliubov. For example, let us rewrite (2.67) as

\[ \frac{dx}{dt} + \psi f(x) = f(x, \delta) \]  

(2.75)

Noting that

\[ \frac{dA}{dt} = \frac{dA}{d\delta} \cos(\omega t + \delta) - \omega A \sin(\omega t + \delta) \]  

one can expand \( f(x, \delta) \) in the Taylor’s series as

\[ f(x, \delta) = f(x, \delta) - \frac{dA}{d\delta} \cos(\omega t + \delta) - \omega A \sin(\omega t + \delta) \]

(2.76)

where \( \text{HOT} \) indicates higher order terms. Because of the slowly varying nature of \( A(t) \) and \( \psi(t) \), \( dA/dt \) and \( d\psi/dt \) are small parameters. Moreover, \( f'(A \cos(\omega t + \delta), -\omega A \sin(\omega t + \delta)) \) indicates differentiation with respect to the second variable, viz.,

\[ -\omega A \sin(\omega t + \delta) \]

In view of the assumption made, one may neglect higher order-terms for weakly nonlinear systems. Also remembering that

\[ \frac{dx}{dt} = -2\omega A \frac{dA}{d\delta} \sin(\omega t + \delta) \]

(2.77)

and comparing (2.75), (2.76) and (2.77) one gets

\[ \frac{dA}{dt} \cos(\omega t + \delta) - \omega A \sin(\omega t + \delta) \]

(2.78)
Therefore, by the method of harmonic balance [9] one derives the following equations from (2.78)

\[ \frac{dA}{dt} = -\frac{\varepsilon}{2\omega A} \frac{1}{\pi} \int f(A \cos (\omega t + \varphi)) \, d\omega \]

and

\[ \frac{d^2 A}{dt^2} = -\frac{\varepsilon}{2\omega A} \frac{1}{\pi} \int f(A \cos (\omega t + \varphi)) \, d\omega \]

(2.79)

and

\[ \frac{d^2 (\omega t + \varphi)}{dt^2} = \frac{\varepsilon}{2\omega A} \frac{1}{\pi} \int f(A \cos (\omega t + \varphi)) \, d\omega \]

(2.80)

These equations are identical with those of Krylov and Bogoliubov except for the second terms on the right hand sides of (2.79) and (2.80).

2.8 Application of the Asymptotic Method

Instead of looking for solutions of the amplitude and phase equations of the nonlinear oscillator, when the nonlinear characteristic have been represented by an nth order polynomial, let us take an example of a practical oscillator and ask the following questions:

1. How is the nature of oscillations affected by the location of the operating point?

2. How is the feedback coupling related to the binning condition for the generation of oscillations?

Consider an oscillator as shown in Fig. 2,3, Fig. 2,4 or Fig. 2,5. The
The Solution of van der Pol's Equation

Refer to Fig. 2.9 and assume that the oscillator is operating around the point \( A \) (cf. Fig. 2.9). In such a condition, the 7th order polynomial representation of the nonlinearity can be approximated by (cf. sections 2.4 and 2.5):

\[
\Sigma C_n x^n = C_1 + C_2 x - C_3 x^3
\]

where

\[
C_1 = \eta f'(0)
\]

\[
C_2 = \eta f''(0) = 1
\]

\[
C_3 = \frac{1}{3!} f'''(0)
\]

Remember \( \eta \) is a factor that depends on the feedback coupling. For a better insight into the operation of such an oscillator, we first consider the nature of oscillation during the initial stage. As pointed out earlier, the solution may be assumed as

\[
x = A \cos(\omega t + \Phi)
\]

with this in mind and utilizing (2.73) and (2.74), the instantaneous amplitude and phase equations are given by

\[
\frac{dA}{d\theta} = \frac{1}{2} [C_1 A \pi - C_2 \pi A^2] - \frac{1}{2} C_3 A [\frac{1}{3} \frac{C_1}{C_2} A^3]
\]

(2.81)

putting

\[
a = A \left[ \frac{C_1}{3C_2} \right]^{1/2}
\]

\[
\frac{da}{d\theta} = \frac{C_1 A}{2A^2} \left( 1 - \frac{a^2}{4} \right)
\]

(2.82)

and

\[
\frac{d\phi}{dt} = -\frac{C_1 A}{2A^2} \left( 1 - \frac{a^2}{4} \right) \frac{dA}{d\theta}
\]

(2.83)
Recall that in deriving the above equation for the instantaneous amplitude and phase of the oscillation the condition of slowly varying parameters has been assumed, i.e.,
\[
\frac{2\pi}{A} \left| \frac{d}{dt} \right| \ll \omega_0 \quad \text{and} \quad \left| \frac{d}{dt} \right| \ll \omega_0
\]  
(2.84)

Therefore, the solution of (2.83) is given by
\[
a(t) = \left[ a(0) \right] \exp \left( -\omega_0 t \right) \]  
(2.85)

where \(a(0)\) is the value of \(a(t)\) at time \(t = 0\), i.e., at the beginning. This indicates that the oscillation will build up spontaneously from any infinitesimally small initial value, provided that \(\omega_1 < 0\).

\[\text{or, for a vacuum tube oscillator}\]
\[
E_0 = \frac{\mu CR}{\mu M - L} > 0
\]
(i.e.,
\[
M > \frac{L}{E_0} + R
\]  
(2.86)

This is what is known as the soft self-excitation. Obviously, the steady-state value of \(\omega\) is \(\omega_1\). Note that if the initial value \(a(0)\) is greater than the final value i.e., \(\omega_1\), the oscillation will continuously decay till the final value is reached.

Once the value of \(A\) is known the instantaneous value of \(\frac{d\omega}{dt}\) can easily be determined. The build-up process of \(A\) and the variation of \(\frac{d\omega}{dt}\) are shown in Fig. 2.14a and Fig. 2.14b. From the figures it is at once seen that during the building up time the frequency of the oscillation is dependent on the instantaneous amplitude of the oscillation, but in the steady state, the frequency of oscillation is independent of the amplitude (cf. dotted curve of Fig. 2.14b). That is, during the build-up process the oscillations are not self-sustained (i.e., frequency of oscillation is independent of the amplitude), but it is so in the steady state. However, the fact is little different, if one takes into consideration the effect of third harmonic distortion in the output.
2.9.1 Effect of Third Harmonic Distortion

When the amplitude of oscillation has reached a fairly large value, the nonlinearity of the device comes into play. As a result, the third harmonic distortion is observed for the cubic type nonlinearity. Therefore, the expected solution of the van der Pol oscillator [5] may be taken as

\[ x = A \cos(\omega_0 t + \gamma) + b \cos(3\omega_0 t + 3\gamma + \theta) \]  

(2.87)

Assuming quasi-stationary state of operation, and linearizing the assumed solution (2.87) to

\[ \frac{d^2 x}{dt^2} + \frac{d}{dt}(C_1 x - C_2 x^3) + \omega_0^2 x = 0 \]

one gets after equating the coefficients of \( \cos 3(\omega_0 t + \gamma) \) and \( \sin 3(\omega_0 t + \gamma) \) to zero

\[ \tan \theta_0 = \frac{8a_4}{3C_1} \]  

(2.88)

and \( 8a_4 \sin \theta_0 = 3C_1 b \cos \theta_0 = 3a_2 a_4 C_1 \frac{e^4}{4} \)  

(2.89)

Since \( a_4 \) is much greater than \( C_1 \), one gets

\[ \theta_0 \approx \pi/2 \]  

(2.90)

and hence

\[ b = \frac{3a_4^8 C_3}{32a_4} \]  

(2.91)
Therefore, the solution of the van der Pol’s equation may be taken as
\[ x = A \cos \theta - b \sin 3\theta \]
where
\[ 6 = a_t + \Psi \]
Thus we write
\[ x^2 = (A \cos 6 - b \sin 30)^2 \]
\[ = A^2 \cos^2 6 - nbA^2 \cos^2 6 \sin 30 \]
\[ + \frac{n(n-1)}{2} A^2 \cos^2 6 \sin 30 \sin 30 + 10 \]

Now, following the method of section 2.7, one gets
\[ \frac{dA}{dt} = \frac{e}{2} \left[ 2A \cos \theta - 3 \sin 3\theta \right] \left( \cos 4\theta - \cos 3\theta + \psi \right) \]
and
\[ \frac{d\Psi}{dt} = -\frac{e}{2} \left[ 2A \cos \theta - 3 \sin 3\theta \right] \left( \sin 4\theta - \sin 3\theta + \psi \right) \]
from which one finds
\[ \frac{ds}{dt} = \frac{e}{2} \left( 3 - a_t^2 \right) \]
and
\[ \frac{d\Psi}{dt} = -\frac{e}{2} \left( 3 - a_t^2 \right) \left( 1 - \frac{3}{4} a_t^2 \right) \frac{ds}{dt} \]
Thus the effect of small third harmonic distortion does not affect the amplitude equation, but it changes the phase equation (cf. 2.83 and 2.99). As a result, the steady state, one finds that

\[ a_s = 2 \]

and \( \omega_s = \text{the frequency of oscillation} \)

\[ \omega_s = \frac{\mu^2}{16a_s} \]  

(2.99)

This indicates that the oscillation is not isoperiodic. The frequency building up is shown by the solid line in Fig. 2.14b.

2.10 Stationary Oscillation and its Stability

Let us now consider the stability of the system when the oscillation has reached its steady state value. Incidentally, the stability of an oscillator means both the amplitude and the frequency stability of the oscillation. The technique of finding the stability of the stationary solution consists in seeking the solution of the amplitude and phase equations when small changes are given in the stationary state of oscillation. When the system is disturbed momentarily from its stable state, let the amplitude and the phase of the oscillation assume \( a_s + \delta a \) and \( \Phi_s + \delta \Phi \) respectively. Once this happens the subsequent behaviour of the oscillation will be completely governed by their respective amplitude and phase equations (2.82 and 2.83), from which it is easy to find the equations for \( \delta a \) and \( \delta \Phi \). Thus

\[ \frac{d}{dt} (\delta a) = -\frac{\mu_0}{2} \left( 1 - \frac{3a}{2} \right) \delta a \]  

(2.100)

and

\[ \frac{d}{dt} (\delta \Phi) = -\frac{\mu_0}{2a_s} \left( 1 + \frac{3a}{2} \right) a_s \delta a \]  

(2.101)

Now remembering that the steady state value of \( a \) is 2, the solution of the above equations are given by

\[ \delta a = \text{Const. exp} (-\mu_0 \delta f) \]  

(2.102)

and

\[ \delta \Phi = -\frac{\mu_0}{a_s} \text{Const. exp} (-\mu_0 \delta f) \]  

(2.103)
Thus it is seen that both the increments in the amplitude and the phase decay exponentially to zero with time at a rate depending upon the nonlinear coupling and the location of the operating point. This indicates that if the steady state oscillation of the van der Pol's oscillator is disturbed, it will return to its steady state condition after showing few cycles of amplitude variation and frequency fluctuation.

2.11 Soft Self-excited Oscillation

Consider the case when the operating point of an oscillator is chosen around the quiescent point (By of Fig. 2.9). In this case, the nonlinear function, and hence, the oscillator equation may be represented by the following equation:

\[ \frac{dB}{dt} - \omega_b^2 B = [C_0 + C_1 x - C_2 x^2 - C_3 x^3] + \omega_b^2 x = 0 \]  

(2.104)

Assuming that the oscillator executes nearly sinusoidal oscillation, i.e.,

\[ x = A(t) \cos(\omega f + \Phi(t)) \]  

(2.105)

It is easily shown (cf. section 2.8) that the asymptotic equation for the amplitude is written as

\[ \frac{dA}{dt} = \frac{\omega_b}{2C_1} \left( C_1 - \frac{3}{4} C_2 A^2 - \frac{1}{2} C_4 A^4 \right) A \]  

(2.105)

Putting

\[ B^t = SC_1 A^t \]  

and \[ \beta = \frac{3C_1}{SC_1^2} \]

Equation (2.106) can be rewritten as

\[ \frac{dB^t}{dt} = -\frac{3C_1}{SC_1^2} (B^t - \beta_i) (B^t - \beta_f) B \]

(2.107)

where \( \beta_i \) and \( \beta_f \) are the roots of

\[ B^t + 2\beta B^t - 8C_1 = 0 \]

i.e.,

\[ \beta_i, \beta_f = -\beta \pm \sqrt{\beta^2 + 8C_1} \]

(2.108)

Remember that

\[ C_1 = \tau f(x_0) = 1 \]  

(2.109)
where \( y \) governs feedback coupling. Fig. 2.15 plots the variation of the steady state amplitude (\( E_s^2 \)) with \( C_y \). It appears from Fig. 2.15, that for all positive values of \( C_y \), the amplitude of oscillation will attain a steady state.

Referring to the amplitude equation one finds that, when the initial amplitude of oscillation is small, the amplitude of oscillation gradually builds up. The growth of the amplitude ceases when the term within the braces goes to zero, and at this stage one gets the amplitude of oscillation in the steady state. The nature of growth of oscillation is here similar to that of the van der Pol's oscillator, except that the rate of growth will be different. In this case one finds that the mode of oscillation is realizable for any value of \( C_y \) greater than zero, i.e., for any value of feedback coupling greater than a critical value corresponding to \( C_y = 0 \) (say \( \eta_0 \)) (cf. 2.109). Phase plane trajectories are shown in Fig. 2.16. It is seen that the phase trajectories, emanating either from small values of \( v \) and \( u \) or larger values of \( u \) and \( v \), approach a closed curve. That is to say, a finite limit cycle is observed for any initial value of \( u \) and \( v \) except \( u = 0 \) and \( v = 0 \). This case of excitation is termed as self-excitation.
Fig. 2.16. Limit cycle operation of a soft self-excited oscillator.

1.12 Hard Self-excited Oscillation

Let us suppose that the oscillator is operating around the quiescent point (C) of Fig. 2.9 or some such point, so that the non-linearity can be represented by the following series

$$\Sigma C_\alpha x^\alpha = C_4 + C_6x + C_{24}x^4 - C_{8}x^8$$  \hspace{1cm} (2.110)

Let us now look into the phase plane plot of the oscillator, having the non-linearity defined above. Consider the oscillator equation

$$\frac{dx}{dt} - \frac{\gamma}{Q} \left[ C_4x + C_6x^6 - C_{24}x^{24} \right] + \alpha_4x^4 = 0.$$  \hspace{1cm} (2.111)

Putting

$$u^4 = SC_4x^4, \quad \beta = \frac{3C_{24}}{(SC_4)^3}, \quad \gamma = \frac{\alpha_4}{Q} \quad \text{and} \quad \tau = \alpha_4t,$$

in (2.111) one gets

$$\frac{du}{dt} - \beta u^{24} + \gamma u^{24} + u = 0.$$  \hspace{1cm} (2.112)

Thus we find that when \(\alpha_4\) is very small (cf. 2.58 and 2.39) and \(C_4\) is negative, \(u\) will decay. However, if the initial value of \(u_0\) (say \(u_0\)) is such that the coefficient of \(\frac{du}{dt}\) is positive, i.e., \(-C_4 + \beta u_0^2 - \gamma u_0^2 > 0\), the oscillation will grow. Therefore, the phase plane equation is written as
66 Phase Lock Theories and Applications

\[
\frac{du}{dt} = v, \quad (2.113)
\]

\[
\frac{dv}{dt} = \gamma (C_1 + \beta u^2 - u^2) v - u, \quad (2.114)
\]

i.e.,

\[
\frac{dv}{dt} = \frac{\gamma (C_2 + \beta u^2 - u^2)}{v} - u. \quad (2.115)
\]

The phase plane plot is shown in Fig. 2.17. Unlike Fig. 2.11, Fig.

![Phase Plane Plot](image.png)

Fig. 2.17. Limit cycle operation of a hard self-excited oscillator.

2.12 and Fig. 2.13, here one finds two limit cycles \(C'\) and \(C\) respectively. The trajectories, starting inside the cycle \(C'\), move towards the origin. On the other hand, the trajectories emanating outside the cycle \(C'\) wind ultimately on the cycle \(C\). The cycle \(C'\) is called an unstable limit cycle and the cycle \(C\) is called a stable limit cycle. Thus, it is seen that such a system requires an impulse greater than a certain threshold value to initiate oscillations. This is called the hard mode of self excitation.

To see this analytically, let us assume the solution of (2.111) in the form

\[
x = A(t) \cos (\omega t + \varphi(t)) \quad (2.116)
\]
one then finds that the amplitude equation is given by
\[ \frac{d^2 A}{d \phi^2} = \frac{m}{Q} \left[ C_1 A + \frac{3}{4} C_2 A^3 - \frac{5}{8} C_3 A^5 \right]. \] (2.117)
In the steady state one has
\[ C_2 A^3 + \frac{3}{4} C_4 A^4 - \frac{5}{8} C_5 A^5 = 0. \] (2.118)
Putting
\[ B^0 = 5C_4 A^4 \quad \text{and} \quad \beta = \frac{3C_2}{(5C_4)^{1/2}}, \]
one finds that the steady state amplitude is given by
\[ B^0 = 2\beta B^0 = -C_4 = 0, \]
which gives
\[ B^0 = 5 \pm (B^0 + 8C_2)^{1/2}. \] (2.119)
The variation of $B^0$ with $C_4$ is shown in Fig. 2.18. From this figure it is seen that real values of $B^0$ are possible even for negative values of $C_4$ provided
\[ |C_4| < \frac{B^0}{8}. \] (2.120)
where
\[ C_4 = \frac{1}{2} \sqrt{\frac{1}{15} - 1}. \]
Before we conclude anything regarding the behaviour of such an
oscillator, we now check the stability of the stationary solution (2.111). We rewrite (2.117) in terms of \( \beta \) as

\[
\frac{d\beta}{dt} = \frac{a_0}{2Q} (8C_1 + 2b\beta - B^2) \tag{2.121}
\]

Thus to check the stability of the mode of oscillation, we put

\[
\beta = \beta_0 + \xi
\]

and write from (2.121)

\[
\frac{d\xi}{dt} = \frac{a_0}{2Q} (8C_1 + 6b\beta_0^2 - 5B^2) \xi
\]

which on comparison with (2.119) yields

\[
\frac{d\xi}{dt} = \frac{a_0}{2Q} (\beta - B^2) \xi \tag{2.122}
\]

Thus, for stability of the stationary solution, \( B^2 \) should be greater than \( \beta \). This is shown in Fig. 2.18. Referring to this figure one finds that when the value of \( C_1 \) lies between 0 and \( Q \), spontaneous generation of oscillations is forbidden, because here, smaller oscillation amplitudes lie in the unstable zone. However, if the initial excitation is such that the oscillation assumes an amplitude that lies above the zone of instability then the oscillation persists. This is known as the hard mode of excitation. Note that when \( C_1 \) lies between 0 and \( Q \), it indicates a smaller value of the coupling coefficient \( (C) \).

Let us now consider the situation when \( C_1 \) is greater than or equal to zero, corresponding to the critical coupling given by

\[
\beta = \frac{1}{2}(0) \tag{2.123}
\]

In this situation one finds from (2.121) that the oscillation grows spontaneously from a very small value and attains a steady value given by (2.119). Once the oscillation has attained the steady state value corresponding to \( F_1 \), one can reduce the value of \( C_1 \), and hence, coupling can be decreased to a value corresponding to that at \( Q \) without quenching the oscillation. This mode of excitation of oscillation is called hard self-excitation.

### 2.13 Remarks

This chapter depicts the picture of nearly sinusoidal oscillations as it is seen through a classical 'telescope'. In two respects it does
present the classical theory. First, it considers three-electrode valve instead of multi-electrode valves or present-day triode-coupling devices. Anyway, the objection that may develop out of this view could be a transient mild shock to the theories presented, as the results derived are still acceptable, provided certain minor adjustments are made in representing the nonlinear transfer characteristics of the active device. Secondly, it makes certain assumptions, as the proponents did in analyzing vacuum tube oscillators. Among them the most important one is the grid current \( i_g \). This is not the grid current that gives the limited type characteristics (except for valves with tungsten cathodes) and avoids the stability of the auxiliary solution. Obviously, this is a pertinent point that needs consideration, but the present chapter does not go into this fact, mainly because \( i_g \) in most of the practical situations automatic biasing circuits or automatic gain control circuits are provided to limit the amplitude of oscillation and \( i_d \) the consideration of grid-current flow complicates the theory to such an extent that the clarity and ease of Nearly sinusoidal oscillations will be deeply immersed in the midst of complexity.

**APPENDIX A**

**Derivation of the Equations for the Negative Resistance Oscillator**

Consider Fig. 2.5b and write

\[
\phi = i_L + i_c, \quad (A-1)
\]

\[
v = L \frac{di_L}{dt} + R_i, \quad (A-2)
\]

and

\[
i_c = C \frac{dv}{dt}. \quad (A-3)
\]

Observe that the current \( i_L \) through the active device is a nonlinear function of the voltage \( v \) across it, i.e., put

\[
i_L = f(v), \quad (A-4)
\]

Hence, using (A-1), (A-2) and (A-3) one finds

\[
L \frac{di_L}{dt} + R_i i_L + L \frac{di_c}{dt} + R_i i_c = v_c + V. \quad (A-5)
\]
Using (A-4), (A-4) and (A-5) one gets
\[ \frac{d^2v}{dt^2} + \frac{1}{C} \int \left[ \frac{BC}{L} \left( v - RF(v) \right) \right] + \frac{1}{LC} \left( v - RF(v) \right) = 0. \quad (A-6) \]

Consider Fig. 2.6b. The nonlinear device impedance (Z) of the Gunn diode [3,4] consists of (i) an equivalent nonlinear resistive part (r) and an equivalent nonlinear reactive part (c). The voltage across these elements are assumed in the following forms
\[ v_r = -\beta_1 j + \beta_1 q^2 + \beta_2 q^3 \quad (A-7) \]
and
\[ v_c = \alpha_1 q + \alpha_2 q^2 + \alpha_3 q^3. \quad (A-8) \]
where,
\[ i = \frac{q}{dt}. \quad (A-9) \]
a_j and \( b_j \) \((j = 1, 2 \text{ and } 3)\) are the constants of nonlinearities of the active device. Referring to Fig. 2.6b and using (A-7) and (A-8), one has
\[ \frac{R}{L} \frac{dq}{dt} + \frac{1}{C} \int \left[ dt + (R + R_c) i - \beta_1 j + \beta_2 q^2 + \beta_3 q^3 + \alpha_1 q + \alpha_2 q^2 + \alpha_3 q^3 \right] \]
\[ + \alpha_1 q + \alpha_2 q^2 + \alpha_3 q^3 = 0. \quad (A-10) \]
It may be noted that \( R, L \) and \( C \) respectively denote the equivalent lumped resistance, inductance and capacitance of the resonator, and \( R_c \) taken into account the equivalent lumped load resistance. Using (A-9) and (A-10) one has
\[ \frac{d^2q}{dt^2} + \frac{\alpha_0}{Q_c} \left[ R + R_c - \beta_1 \right] \frac{dq}{dt} + \frac{\beta_2}{R_c} \frac{dq}{dt} \left[ \frac{dq}{dt} \right] - \frac{\beta_3}{R_c} \left( \frac{dq}{dt} \right)^2 \frac{dq}{dt} \]
\[ + \alpha_1 q + \alpha_2 q^2 + \alpha_3 q^3 = 0. \quad (A-11) \]
where,
\[ \alpha_0 \frac{1}{Q_c} \text{ and } \frac{Q_c}{R_c}. \]

**APPENDIX D**

Derivation of Equations of Feedback Type Oscillators

Refer to a classical oscillator having the circuit configuration as shown in Fig. 2.7. One writes
\[ i_r = i_i + i_c, \quad (B-1) \]
where \( i_2 \) and \( i_c \) are the instantaneous values of the respective currents through the inductive and capacitive branch of the dual circuits incorporated in the oscillator. \( i_p \) denotes the total plate current which may be written as

\[
i_p = i_2 + i_c
\]

where \( i \) is the amplification factor of the tube,

\[
v_p = V_F + v_c = V_F + \frac{V_p}{\alpha}
\]

and

\[
v_c = V_G + v_p
\]

where the symbols have their usual significance. That is, a lower case letter suffixed with an upper case one denotes the total value of the parameter consisting of a d.c. and an a.c. term. A lower case letter suffixed with a lower case one indicates an alternating component, whereas an upper case letter suffixed with an upper case one equals a direct component term. The suffixes \( F \) and \( G \) denote respectively the corresponding variables for the plate and the grid of the tube.

Considering Fig. 2.7, one has the following relations

\[
\begin{align*}
\eta &= V_{pp} - L \frac{di_p}{dt} - RC_i \\
\psi &= V_{gg} + M \frac{di_c}{dt}
\end{align*}
\]

and

\[
\frac{1}{C} \int icdt = L \frac{di_c}{dt} + R_i \psi
\]

where \( M \) is the mutual inductance of the coils.

Since there is no resistance in the grid circuit, \( V_0 \) equals \( V_{gg} \).

Now taking the help of equations (B-1) and (B-7), one obtains

\[
i_p = LC \frac{di_p}{dt} + RC \frac{di_c}{dt} + i_c
\]

Using (B-6), equation (B-8) may be rewritten as

\[
\frac{d^2v_c}{dt^2} + \frac{R}{LC} \frac{dv_c}{dt} + \frac{1}{LC} (V_0 - V_{gg}) = A_0 \frac{di_c}{dt}
\]
where \( \omega_0^2 \left( \frac{1}{L} \right) \) denotes the resonant frequency of the tank circuit.

Therefore, the differential equation for the varying components of plate current and the grid voltage is given by

\[
\frac{d^2 v_p}{dt^2} + \frac{R}{L} \frac{dv_p}{dt} + \omega_0^2 v_p = M \frac{dv_e}{dt}.
\]  
(B-10)

From (B-5) and (B-6) one finds that

\[
v_e = V_{pp} - \frac{L}{\mu} (v_0 - V_{pp}) - R i_e.
\]  
(B-11)

Noting that 'i_e' consists of an alternating component 'i_e^r' and a direct component of current i_e, one gets

\[
v_e = \frac{V_{pp} - L i_e^r - \frac{\mu}{\mu} i_e^d + \frac{\mu}{\mu} V_{pp}}{\mu} + (v_0 + v_e)
\]  
(B-12)

Thus from (B-2) and (B-12), one has

\[
i_e = f \left[ v_e - \left( \frac{\mu}{\mu} i_e^r + \frac{\mu}{\mu} V_{pp} \right) + v_e \right]
\]  
(B-13)

where

\[
V_e = V_{pp} - \frac{L}{\mu} i_e^r + V_{pp} e
\]  
(B-14)

It may be noted that 'V_e' denotes the location of the operating point.

Usually, the value of 'R' is small in practice. Thus expanding the function \( f \left( \frac{v_e}{\mu} + v_e \right) \) by Taylor's series, (B-13) looks like

\[
i_e = f \left( V_e + \frac{\mu}{\mu} - \frac{L}{\mu} i_e^r - v_e \right) - \frac{R}{\mu} i_e^r.
\]

\( \frac{dV_e}{dt} \) is the ac conductance of the tube at the operating point (V_e). Therefore

\[
i_e = f \left( V_e + \frac{\mu}{\mu} - \frac{L}{\mu} i_e^r \right) - \frac{R}{\mu} i_e^r
\]  
(B-15)

where \( r_p \) is the ac plate resistance of the tube at the operating point (V_e).

Noting that \( i_e = M \frac{dv_e}{dt} \), and using (B-10) and (B-15), one has the
following equation,
\[
\frac{dv_y}{dt} + \frac{v_y}{Q} \frac{dQ}{dt} (v_y - F(v_x)) + \omega^2 \left( 1 + F'(v_y) \right) v_y = 0
\]  \hspace{1cm} \text{(B-16)}

where
\[
Q = \frac{v_x}{F} \text{ and } R(v_y) = \frac{M}{E} f \left( v_y + \frac{v_y^{1/M-1} - v_x}{p} \right)
\]

For the transistor oscillator equation, refer to Fig. 2.8 and observe that
\[
ic = LC \frac{dv_y}{dt} + RC \frac{d^2v_y}{dt^2} + i_e \]  \hspace{1cm} \text{(B-17)}

and
\[
V_{ss} - v_y = M \frac{dQ}{dt} - L \frac{dv_y}{dt}
\]  \hspace{1cm} \text{(B-18)}

where the variables with suffix 'ss' signify the corresponding values for the base of the transistor. Since \( M \) is much greater than \( L \), \( (B-18) \) approximates to
\[
V_{ss} - v_y = M \frac{dQ}{dt}
\]  \hspace{1cm} \text{(B-19)}

Putting
\[
v_y = V_s - V_{ss}
\]  \hspace{1cm} \text{(B-20)}

one has,
\[
v_b = -M \frac{dQ}{dt}
\]  \hspace{1cm} \text{(B-21)}

Using \( (B-17) \) and \( (B-21) \) one finds,
\[
\frac{d^2v_y}{dt^2} + R \frac{dv_y}{dt} + \frac{1}{LC} v_y = -M \frac{dQ}{dt}
\]  \hspace{1cm} \text{(B-22)}

Denoting the functional relation between the emitter current and the base voltage as
\[
i_b = -f(V_{ss} + v_y)
\]  \hspace{1cm} \text{(B-23)}

and noting \( a \) to be collector-emitter short-circuit current amplification factor, one has
\[
\frac{d^2v_y}{dt^2} + R \frac{dv_y}{dt} + \frac{1}{LC} v_y = M a \frac{dQ}{dt} f(V_{ss} + v_y)
\]  \hspace{1cm} \text{(B-24)}
Therefore,

\[ \frac{d^2 y}{dt^2} + R \frac{dy}{dt} - \frac{M_e}{L} \frac{d}{dt} \left( V_{z3} + \gamma_3 \right) + \frac{1}{C_e} \gamma_3 = 0 \]  

(B-25)

Putting

\[ o_3 = \frac{1}{C_e} V - \frac{m_e L}{R} \text{ and } N(\gamma_3) = \frac{M_e}{C_e} (V_{z3} + \gamma_3) \]

one obtains from (B-25)

\[ \frac{d^2 \gamma_3}{dt^2} + \frac{m_e L}{R} \frac{d\gamma_3}{dt} - \frac{d}{dt} N(\gamma_3) = \frac{1}{C_e} \gamma_3 = 0 \]  

(B-26)

REFERENCES


CHAPTER 3

INFLUENCE OF A SINUSOIDAL SIGNAL ON CLASS-A OSCILLATORS

3.1 Introduction

A sinusoidal signal of frequency \( \omega_f \), acting on a nearly sinusoidal oscillator of a free-running frequency \( \omega_0 \), affects its behaviour in many respects depending on the strength and frequency of the forcing signal. Consider the circuit of Fig. 3.1a to which a forcing signal is injected. In such a case, one observes the well-known phenomenon of beats having a beat frequency that depends on the frequency and the strength of the forcing signal in relation to those of the local oscillation [1,2]. A typical plot depicting the variation of beat angular frequency is shown in Fig. 3.1b. It shows that the beat angular frequency decreases linearly with \( \omega_0 \), up to a value \( \omega_{0b} \) after which it suddenly drops to zero and remains zero up to \( \omega_{0h} \) from where it suddenly jumps to a finite value. Thus, within this zone, \( \omega_{0h} - \omega_{0b} \), the instantaneous frequency of the oscillator is identical with that of the external signal. That is to say, within this zone, called the synchronization range, the local oscillator loses its identity and obeys the command from the forcing signal; and the oscillator is said to be phase locked to the synchronizing signal. Outside the synchronization range, the oscillator is said to be unlocked. It is interesting to observe that the average frequency of the oscillator in the unlocked state does not become equal to the free running frequency of the oscillator, and it approaches the free running one when the frequency of the synchronizing signal is far away from the free running frequency (cf. Fig. 3.1c).

Obviously, the tendency of the local oscillator to fall in synchronism with the forcing signal cannot be explained from linear analy-
Fig. 3.1 (a) Injection synchronization in an oscillator, (b) its locking characteristics, and (c) its tracking characteristic.

sis of the circuit which predicts a gradual decrease of the beat frequency. On the other hand, nonlinear circuit analysis must be used to explain the phenomenon of synchronization. For the moment, let us not go into the extremely complicated law that dictates the variation of the frequency and amplitude of the local oscillator. Instead, let us refer to Fig. 3.1b and Fig. 3.1c and conclude the following: (i) the frequency of the local oscillator is almost unaffected if the frequency of the forcing signal differs considerably from that of the local oscillator, (ii) the centre frequency of the oscillator is "pulled" towards that of the forcing signal when \( a_1 \) approaches that of the oscillator, and (iii) at a certain value of \( a_1 \), the centre frequency of the oscillator is "pulled-up" to the external signal frequency. In the following we give a simple explanation [4] of the phenomenon of synchronization.
3.1.1 PHYSICAL-ANALYTICAL APPROACH

Let us examine the behaviour of a self-excited class-A oscillator in response to a sinusoidal signal. Class-A signifies the operation of the active device in the class-A condition without the flow of current to the input port. For example, consider the oscillator circuit shown in Fig. 3.1b and observe its behaviour at a particular instant when an external signal $E \cos \omega t$ is injected in the plate circuit. As a result of this, a signal, say, $E \cos \omega t + \varphi$, will appear at the grid circuit. Because of the inherent nonlinearity provided by the grid voltage and the plate current characteristics, there will be a nonlinear interaction between the external input $E \cos \omega t$ and the self-excited oscillation as observed in the grid circuit, say, $A\cos (\omega t + \varphi)$. It is expected that as soon as the external signal is injected, the oscillation will no longer be represented as $A\cos (\omega t + \varphi)$, but it is to be denoted as $A\cos (\omega t + \varphi)$, because the presence of the external signal modifies the operating characteristics of the oscillator. It is known that the dependence of the plate current on the grid voltage may be represented by the following functional relation:

$$i_p = \sum C_v \alpha f_s$$

(3.1)

where $C_v\alpha$ are constant and

$$r_p = E \cos \omega t + A \cos (\omega t + \varphi)$$

(3.2)

Taking the plate current versus the grid voltage characteristics as

$$i_p = C_v + C_v \mu^2 + C_v - C_v \mu^2$$

(3.3)

it can be easily shown that the fundamental component of plate current close to $\omega t$ is given by

$$i_p = -A C_v \left[ \cos (\omega t + \varphi) + \frac{E}{A} \cos \omega t \right]$$

$$- \frac{3}{4} C_v \mu^2 \cos (\omega t + \varphi) + 2 \frac{E}{A} \cos \omega t$$

$$+ 2 \left( \frac{E}{A} \right)^2 \cos \omega t$$

$$= \frac{3}{4} C_v \mu^2 \left( \frac{E}{A} \right)^2 \cos (\omega t - \varphi)$$
Now assume that the strength of the external signal is small in comparison with that of the oscillator. This is usually the case in most of the practical situations. Thus one finds that (3.4) reduced to

\[
1_{1P} \approx C_A \left[ 1 - \frac{3}{4} \frac{C_A}{C_A} A^2 \cos (\omega_0 t + \Psi) + \frac{1}{4} \frac{C_A}{C_A} A^2 \cos \theta \right]
\]

representing the phase difference between the two signals \( A \cos (\omega_0 t + \Psi) \) and \( \cos (\omega_0 t + \Psi) \). One gets the following expressions for the quadrature and in-phase components of \( \phi \):

\[
(\phi_0) = C_A \left[ \frac{1}{4} \frac{C_A}{C_A} A^2 \right]
\]

and

\[
(\phi_0) = \frac{1}{2} \frac{C_A}{C_A} A^2 \cos \Psi \sin (\omega_0 t + \Psi)
\]

where \((\phi_0)\) and \((\phi_0)\) are respectively the in-phase and quadrature components of \( \phi \).

That is to say, in the presence of an external signal the transconductance of the tube can be imagined to consist of an in-phase and a quadrature component. The in-phase component of the transconductance is (cf. 3.7)

\[
\frac{g_A}{Z} = C_A \left[ 1 - \frac{3}{4} \frac{C_A}{C_A} A^2 \cos \theta \right]
\]

and the quadrature component of the transconductance is

\[
\frac{g_0}{Z} = \frac{1}{4} \frac{C_A}{C_A} A^2 \sin \Psi
\]
Thus \( \phi \), the in-phase component of the transconductance, modulates the gain parameter of the oscillator, i.e., the instantaneous amplitude of the oscillation, whereas \( \phi_0 \), the quadrature component of the transconductance, modulates the instantaneous frequency of oscillation. Now if the frequency difference is small and the strength of the synchronizing signal is adequate, it is expected that the phase difference \( \phi \) attains a steady state value and the oscillator falls in synchronism with the external signal.

Let us suppose that the synchronizing signal is slightly out of tune with respect to that of the free-running oscillator. Now refer to the circuit and observe the phase difference \( \phi_r \) between the plate current and the voltage across the tank circuit at a frequency \( \omega \). Obviously, this is given by [5]

\[
\phi_r = -\arctan \left( \frac{\omega}{\omega_0} \right)
\]

(1.11)

If \( \omega \) is close to \( \omega_0 \), one has

\[
\phi_r = -\arctan \left( \frac{2(\omega - \omega_0)}{\omega_0} \right)
\]

(3.12)

where \( \omega_0 \) is the resonant frequency of the tank circuit which is nearly equal to the free running frequency of the oscillator. Phase relation between \( \phi_r \) and \( \phi_0 \) is shown in Fig. 3.2. If the resistance of the tank circuit is small, the feedback voltage to the grid circuit

\[
\phi = A \cos (\omega t + \Phi)
\]

Fig. 3.2. The phasor diagram of the synchronized oscillator.
is related to the plate voltage swing $\tau_P$ by the following approximate relation

$$\tau_P \approx \frac{L}{M} \frac{dv}{dt} = \frac{L}{M} \frac{dE}{dt} \quad (3.13)$$

i.e., they are in phase which is shown in the phase diagram. Note that $\tau_P$ is added to the external signal $\tau = E \cos \omega t$. This is also shown in Fig. 3.2. The vectorial sum of $\tau_P$ and $\tau$ should be along $L$ as the plate current and the net grid voltage i.e., $\tau_P + \tau = \tau$ is always in phase. Therefore, referring to Fig. 3.2, one gets

$$AB \sin \Phi = OB \sin \theta \quad (3.14)$$

where

$$AB = E \quad (3.15)$$

$$OB = A + E \cos \Phi \quad (3.16)$$

Since $E$ is small

$$OB = A + E \cos \Phi \quad (3.17)$$

Since $\Phi = \theta$, and $w = \omega_0 + \frac{d\theta}{dt}$

$$\theta = -\frac{2Q}{\omega_0^2} \left[ \omega_0 + \frac{d\theta}{dt} - \omega_0 \right] \quad (3.18)$$

one has from (1.12) through (3.11),

$$\frac{d\theta}{dt} = \frac{\omega_0 - \omega_0}{2} \left[ E \sin \Phi \right] \quad (3.20)$$

or

$$\frac{d\theta}{dt} = \omega_0 - \omega_0 - E \sin \Phi \quad (3.21)$$

where

$$\theta = \frac{\omega_0}{2} \frac{E}{A} \quad (3.22)$$

If $(\omega_0 - \omega_0)$ is greater than $K$, one gets from (3.22) [6]

$$\Phi = 2 \tan^{-1} \left[ \frac{\sqrt{X^2 - 1}}{X} \tan \frac{\sqrt{X^2 - 1}}{2} \left( \frac{1}{K} + \frac{1}{2} \frac{E}{A} \right) \right] \quad (3.24)$$
where
\[ \chi = (\alpha_1 - \alpha_0)/K \]  
and \( \alpha_0 \) is a constant. Putting
\[ \theta = \sqrt{\frac{K^2}{\tau^2} - 1}, (t + t_0) \]
\[ \beta = \tan^{-1} \sqrt{\frac{K^2}{\tau^2} - 1} \]  
(3.27)
Differentiating (3.34) one finds
\[ \frac{d\theta}{dt} = K (\theta^2 - 1), \sec^2 \theta \frac{d\theta}{dt} = \sec^2 \theta/2 \]  
(3.28)
Using the identity
\[ \sec^2 \theta/2 = 1 + \tan^2 \theta/2 \]  
(3.29)
and combining (3.24), (3.28) and (3.29) one gets [4]
\[ \frac{d\theta}{dt} = \frac{K \theta^2 - 1}{\theta^2 + \cos (2\theta - \beta_0)} \]  
(3.30)
Put
\[ t' = X - \sqrt{\theta^2 - 1} \]  
(3.31)
and use the identity
\[ \frac{1}{\theta^2 + \cos \theta} = \frac{1}{2} (\cos \theta - (\theta^2 - 1)) \cos \theta + \frac{1}{2} (\theta^2 - 1)\sin \theta \]  
(3.32)
where
\[ \beta = 2\theta - \beta_0 \]  
(3.33)
Note that the real part of the left hand side of (3.32) can also be written as
\[ K \left[ 1 + \frac{1}{2} \frac{1}{\theta^2 + \cos 2(\theta - \beta_0)} \right] \]  
(3.34)
Therefore, from (3.33), (3.34) and (3.30) one gets
\[ \frac{d\theta}{dt} = \frac{K \sqrt{\theta^2 - 1} \left[ \frac{1}{2} (\theta^2 - 1) - (\theta^2 - 1)\cos (2\theta - \beta_0) \right]} \]  
(3.35)
The average value of the oscillator frequency is the presence of the synchronizing signal is thus given by
\[ \langle \omega_0 \rangle = \alpha_1 + \frac{d\theta}{dt} = \alpha_2 - \langle \delta \rangle \]  
(3.36)
where \( \phi = \frac{dp}{dt} \)

and, (\( \cdot \)) denotes the average value.

Hence from (3.18) and (3.35) one can show that

\[
(\bar{\omega}_s) = \omega_0 + \left[ (\omega_1 - \omega_0) + \sqrt{(\omega_1 - \omega_0)^2 + \vec{E}_0^2} \right] (3.57)
\]

Thus depending upon the sign of \((\omega_1 - \omega_0)\), the average oscillator frequency will be lower or higher than the free-running frequency. It is seen that the lower the value of \((\omega_1 - \omega_0)\) is in relation to \(E_0\), the higher will be the shift \((\bar{\omega}_s) \approx \omega_0\). This means that the centre frequency of the oscillator will be 'pulled' towards the frequency of the external signal. At a certain value of the difference frequency, the pull may be so large that the centre frequency of the oscillator becomes equal to that of the external emf. As soon as this becomes the situation, the oscillator loses its identity and its frequency of oscillation becomes entrained by the external signal. We may say that synchronization has been established between the local oscillation and the external signal. The difference of frequency \((\omega_1 - \omega_0)\), at which synchronization sets in, can be had by using the concept of frequency pull-on, i.e., (cf. Fig. 3.1a),

\[
\frac{d}{d\Lambda} (\omega_1 - \omega_0) = \infty \quad (3.38)
\]

where,

\[
\Lambda = (\omega_1 - \omega_0).
\]

Thus from (3.37) and (3.38) one has the one-sided locking range

\[
\omega_0 - \omega_0 = \Phi \quad (3.39)
\]

3.2 Analytical Method

The above discussion explains the phenomenon of synchronization in simple physical terms. From what has been said it is clear that entrainment is a nonlinear phenomenon. It is happening due to the nonlinearity provided by the plate current and grid voltage characteristics of the tube. Therefore, the phenomenon will not depend upon the location of the oscillator circuit at which the external signal is injected into the loop. Let us refer to Fig. 3.1b and write the nonlinear equation of the oscillator as (cf. section 3.4).
54 Phase Lock Theory and Applications

\[ i_p - i_s = L \frac{d^2 i_p}{dt^2} + R C \frac{di_p}{dt} + i_s \]

(3.40)

Here \(i_s\) indicates the signal component of current injected into the plate circuit. Therefore, writing (3.40) in terms of the output of the oscillator, one has

\[ \frac{dI}{dt} \left( i_i - i_s \right) = \frac{d^2 i_i}{dt^2} + \frac{E}{L} \frac{di_i}{dt} + \frac{r_o}{L} \]

(3.41)

where

\[ r_o = \frac{h}{\frac{dI}{dt}} \]

(3.42)

Thus one can rewrite (3.41) as

\[ \frac{d^2 i_i}{dt^2} + \frac{r_o}{L} \left( \frac{di_i}{dt} - \frac{E}{L} \frac{di_i}{dt} \right) + \frac{h}{L} \frac{di_i}{dt} = - \frac{h}{L} M \frac{dI}{dt} \]

(3.43)

\[ \frac{d}{dt} = \frac{h}{L} \left( 1 + \frac{R}{r_o} \right) \]

Assuming

\[ I = 1 \cos \omega t \]

one can write the right hand side of (3.43) as

\[ -h \omega^2 \frac{dI}{dt} = - \frac{M}{L} \frac{di_i}{dt} \]

\[ - \frac{M}{L} \omega^2 \frac{di_i}{dt} = \frac{R}{r_o} \frac{di_i}{dt} \]

since \( r_o \gg R \) and \( 1/Q = R/L \)

one gets

\[ -M \omega^2 \frac{dI}{dt} = \frac{h}{L} \left( \frac{M}{L} \frac{di_i}{dt} \right) \]

(3.44)

It is not difficult to appreciate that \( \frac{M}{L} \left( \frac{L}{CR} \right) \) (= \( E \)) denotes an equivalent value of the peak synchronizing voltage injected in the grid circuit. Thus we rewrite (3.43) as

\[ \frac{d^2 i_i}{dt^2} - \frac{h}{Q} \left( \frac{di_i}{dt} \right) \frac{d}{dt} \left( E \cos \omega t \right) \]

\[ = h \omega^2 \frac{di_i}{dt} - \frac{h}{Q} \left( E \cos \omega t \right) \]

(3.44)
Referring to the other oscillator circuit, it is not hard to show that in the presence of the external signal, the oscillator equation can, in general, be written as

\[
\frac{d^2x}{dt^2} - \frac{\omega_0}{Q} \left( x - \eta (x_0 + x) \right) + \omega_0^2 x = \frac{\omega_0}{Q} E \sin \omega t
\]  

(3.45)

3.3. Entrainment of a van der Pol Oscillator

Choose the operating point in such a way that the non-linearities can be represented by a cubic type relation (see Section 2.9), and write

\[
\frac{d^2x}{dt^2} - \frac{\omega_0}{Q} \left( x - C_3 x^3 \right) + \omega_0^2 x = \frac{\omega_0}{Q} \sin \omega t
\]  

(3.46)

Before embarking upon the method of solving (3.46), let us assume that (i) the strength of the external signal is small compared to that of the free-running van der Pol oscillator, having a small value of \( \epsilon = C_3 \omega_0 / Q \) and (ii) the difference of frequency between the external signal and the free running oscillator is not large. As a result of these assumptions, it is evident that the variations of the instantaneous frequency and amplitude of the forced oscillator can be regarded as a slow process. Thus we assume the solution of (3.46) is of the form

\[ x(t) = A(t) \cos (\omega t + \Phi(t)) \]  

(3.47)

where \( A(t) \) and \( \Phi(t) \) are slowly varying functions of time.

That is, we assume

\[
dA \approx 0
\]  

(3.48)

\[
d\Phi \approx 0
\]  

(3.49)

\[
\frac{dx}{dt} \approx 0
\]  

(3.50)

\[
\left( \frac{d^2x}{dt^2} \right) \approx 0
\]  

(3.51)

and

\[
\frac{d\Phi}{dt} \approx \omega
\]  

(3.52)
Thus we write
\[
\frac{dx}{dt} = -A A (\omega_0 + \Psi) - \omega_0 A \sin (\omega_0 t + \Psi) \tag{3.53}
\]
\[
\frac{d\Psi}{dt} = -2\omega_1 \frac{DA}{dt} \sin (\omega_0 t + \Psi)
- \frac{A}{2} \left(\omega_0^2 + 2\omega_0 \frac{d\Psi}{dt}\right) \cos (\omega_0 t + \Psi) \tag{3.54}
\]

Recalling the identity
\[
x_t = \frac{1}{4} A^4 \left[\cos 3(\omega_0 t + \Psi) + 3 \cos (\omega_0 t + \Psi)\right]
\]
and putting
\[
\Phi = -\Psi \tag{3.55}
\]
It is easy to show from (3.46), (3.53), (3.54) and (3.55) with the help of harmonic balance [7] that
\[
\frac{dA}{dt} = -\frac{\omega_0}{2Q} \left(\frac{1}{4} A^4 \right) A + \frac{\omega_0 E}{2Q} \cos \Phi \tag{3.56}
\]
\[
\frac{d\Phi}{dt} = -\frac{\omega_0}{2} \left(\frac{\omega_0 - \omega_0}{\omega_0} - \frac{\omega_0}{2Q} \right) A + \frac{\omega_0 E}{2Q} \sin \Phi \tag{3.57}
\]
Equations (3.56) and (3.57) give the amplitude and phase variations of the forced oscillator.

Putting
\[
A_0 = \sqrt{\frac{4E}{12Q}} \tag{3.58}
\]
\[
a = A/A_0 \tag{3.59}
\]
\[
q = \frac{1}{\omega_0} \tag{3.60}
\]
and
\[
F = \frac{E}{C_2 A_0} \tag{3.61}
\]
we rewrite (3.56) and (3.57) as
\[
\frac{dA}{dt} = \frac{\omega_0}{2Q} (1 - a^2) A + \frac{\omega_0 E}{2Q} F \cos \Phi \tag{3.62}
\]
and
\[
\frac{dp}{dt} = \frac{a_0}{2} \left( e_{a_0} - e_{a_2} \right) - \frac{a_0}{2q} E \sin \varphi
\]  
(3.63)
Assuming for the moment that the amplitude fluctuation is negligible in an under-driven oscillator (i.e., \( a = 1 \), \( A = A_d \)) one finds that (3.63) reduces to
\[
\frac{dp}{dt} = \Omega - K \sin \varphi
\]  
(3.64)
where
\[
\Omega = \frac{a_0}{2} \left( e_{a_0} - e_{a_2} \right)
\]  
(3.65)
and
\[
K = \frac{a_0}{2q} E = \frac{a_0}{2q} \frac{E}{A_d}
\]  
(3.66)
Equation (3.64) can be easily solved to show [9] that
\[
\varphi = \text{arc tan} \left[ \frac{1}{X} + \frac{(X^2 - 1)^{1/2}}{X} \tan \left( \frac{K}{2} (X^2 - 1)^{1/2} (t + \omega) \right) \right]
\]  
(3.67)
for \( X > 1 \), i.e., \( \Omega > K \)
and
\[
\varphi = 2 \text{arc tan} \left[ \frac{1}{X} - \frac{1}{X} \frac{1 - X^2 (l + k) \sin (-k_d)}{X (1 - k) \cos (-k_d)} \right]
\]  
(3.68)
for \( X < 1 \), i.e., \( \Omega < K \)
where
\[
X = \frac{\Omega}{K}
\]
and
\[
k = -\tan \left( \frac{\pi}{2} (a/2) \right) + 1 - (1 - X^2)^{1/2}
\]
and
\[
k_d = K (1 - X^2)^{1/2}
\]
Note that \( \eta \) and \( X \) are constants of integration.
From (3.65) and (3.67) we conclude that (i) when \( X > 1 \), \( \Omega > K \) then unity, the phase difference \( \varphi \) attains a steady value, meaning thereby that the frequency of the forced oscillator becomes equal to the frequency of the synchronizing signal and (ii) when \( X \) is greater than unity the phase difference does not attain steady state.
3.3.1 Locked Behaviour

In this section we consider the behaviour of the oscillator when \( \Omega < X \) and the amplitude variation is taken into consideration. If under this condition, entrainment occurs, the instantaneous frequency of the oscillator should be equal to that of the forcing signal. Moreover, if the entrainment is stable, any variation in \( A \) and \( \phi \) should damp out.

Referring to (3.62) and (3.63) we find

\[
\frac{d\phi}{dt} = \frac{(1 - d^2) a_1 + F \cos \psi}{F X^2 - \sin \phi_0} \tag{3.69}
\]

Obviously, the van der Pol oscillator will be entrained if (3.69) has a singularity where

\[ (1 - d^2) a_1 + F \cos \psi = 0 \tag{3.70} \]

and

\[ X - \sin \phi_0 a_1 = 0 \tag{3.71} \]

i.e.,

\[ (1 - d^2) a_1 + F \cos \psi = \frac{F a_1}{a_2} \tag{3.72} \]

Under the condition of entrainment, the amplitude of the oscillator is given by (3.72), which is a cubic equation in \( a \). If positive real solutions of (3.72) are obtained, entrainment is possible, but it may not be stable. On the other hand if (3.72) has complex solutions, then entrainment is not possible at those values of the amplitudes.

Because of the difficulty in finding an analytical solution of (3.72) which is a cubic equation in \( a \), usually graphical method is adopted. Fig. 3.3 plots a variation of \( a \) with \( \frac{d\phi}{dt} \).

To the goal of finding stable solutions, [9, 10, 11] we replace \( \psi \) and \( \eta \) in (3.62) and (3.65) respectively by \( a_1 + \mu \) and \( \psi_0 + \tau \). Here, \( \mu \) and \( \tau \) are arbitrarily small quantities. Write the incremental equations as

\[
\frac{d\mu}{dt} = \frac{\omega_0}{2} \sin \phi_0 a_1 + \frac{\omega_0}{2} \sin \psi_0 \tag{3.73}
\]

\[
\frac{d\tau}{dt} = \frac{\omega_0}{2} \sin \phi_0 a_1 + \frac{\omega_0}{2} \sin \psi_0 \cos \phi_0 \tag{3.74}
\]

where \( \phi_0 \) and \( a_1 \) are given by
\[ (1 - a^2) u_\nu + F \cos \varphi_\nu = 0 \quad (3.75) \]
\[ \varphi \left( \frac{u_\nu}{v_\nu} - \frac{v_\nu}{u_\nu} \right) - \frac{r}{d} \sin \varphi_\nu = 0 \quad (3.76) \]

From which one gets the characteristic equation as

\[ s - \left( 1 - 3b^2 \right) \frac{c^2}{s^2} \frac{v_\nu}{u_\nu} \sin \varphi_\nu - \frac{b}{d} \sin \varphi_\nu \]
\[ + \frac{b}{d} \frac{v_\nu}{u_\nu} \cos \varphi_\nu = 0 \quad (3.77) \]

This can be expanded to form a quadratic equation of the form

\[ s^2 + bs + c = 0. \quad (3.78) \]

For stability it is necessary that the roots of the equation should have negative real parts. This is guaranteed when \( b > 0 \) and \( c > 0 \). Using these conditions along with (3.77) and (3.78) it is readily shown that for stability, it is necessary and sufficient to have

\[ a > 0.5 \quad (3.79) \]
\[ (1 - a^2) (1 - 3b^2) + \varphi \left( \frac{u_\nu}{v_\nu} - \frac{v_\nu}{u_\nu} \right) > 0 \quad (3.80) \]

The stability boundary, obtained from (1.79) and (3.80), is shown by the hatched boundary in Fig. 3.3. Equation (3.80) represents an ellipse.

The response curves, as shown in Fig. 3.3, exhibit peculiarities for certain values of \( a \) and \( b^2 \), which are not common to linear systems. Here more than one stable value of oscillator amplitude is observed for a certain strength of the synchronizing signal. Consider the magnified portion of the response characteristics as shown in Fig. 3.3. For example, if one gradually increase the detuning of the synchronizing signal (point A), the oscillator amplitude decreases following the path A B C to the point C. A slight increment in detuning at the point C causes an abrupt fall in the oscillator amplitude to the point D, and thereafter follows the path D E. The fall of the amplitude from C to D is known as the jump phenomenon. After having reached D, if one decreases the frequency of the synchronizing signal, the amplitude of the oscillator increases and follows the path D E to the point E. A further lowering of detuning causes another jump from the point E to the point D. This shows a sort of hysteresis in the system.
It is interesting to note that the areas representing the zones of hysteresis in Figs. 3.3 (b) and 3.3 (c) are not equal. This comes in because of the asymmetric response characteristic of the tank circuit of the oscillator. It is, therefore, worthy to mention that a proper choice of oscillator parameters enables one to get rid of either the jump phenomenon or hysteresis on one side of the centre frequency, while keeping the other on the other side for a given strength of the synchronizing signal [9].

To find the synchronization range, \( \frac{dv}{dt} \) and \( \frac{da}{dt} \) are equated to zero, whence we get from (3.62) and (3.63)

\[
a^2 - a - F \cos \phi = 0 \tag{3.81}
\]

and

\[
\Omega - \frac{dv}{dt} = F \sin \phi \tag{3.82}
\]

Again by putting
\[ K = \frac{\Theta}{2Q} F = \frac{\Theta}{2Q} A \sin \phi \]  
(3.83)

we rewrite (3.82) as

\[ \Omega = K \sin \phi \]  
(3.84)

To find the locking range, i.e., the maximum frequency error up to which locking is maintained, we now maximize \( \Omega \) by changing \( \phi \) and hence \( \psi \). A plot of \( \Omega F \) vs. \( \phi \) is shown in Fig. 3.4 from which the locking ranges for two values of \( F \) are given by

![Graph](image)

Fig. 3.4: Evolution of the locking range when amplitude variation of an oscillator is considered.

\[ \frac{\Theta}{K} = 1.055 \quad \text{for} \quad F = 0.2 \]  
(3.85a)

and

\[ \frac{\Theta}{K} = 1.022 \quad \text{for} \quad F = 0.4 \]  
(3.85b)

If the strength of the forcing signal is not large, then \( \psi \) is close to unity. Hence from (3.81) one finds that \( \psi \) is approximately given by $1/2$. (Variation of \( \psi \) with \( F \) and \( \phi \) is more important for the calculation of locking characteristics.)
\( \theta = \frac{1}{4} + \frac{E}{2} \cos \varphi \)  

(3.85)

Hence (3.84) can be written as

\[ \frac{\Omega}{K} = \frac{\sin \varphi}{1 + \frac{E}{2} \cos \varphi} \]  

(3.86)

Maximizing the value of \( \Omega/K \) with respect to \( \varphi \), one finds [5]

\[ \frac{\Omega}{K} = \frac{\pm 1}{\sqrt{1 - \frac{E^2}{4}}} \]  

(3.87)

which agrees well with the results of (3.85a) and (3.85b).

The value of \( \varphi \) for which the maximum range appears is given by,

\[ \cos \varphi = -\frac{E}{2} \]  

(3.88)

Putting

\[ K_e = \frac{K}{\sqrt{1 - \frac{E^2}{4}}} \]  

(3.89)

and inserting the value of \( \Omega \) from (3.65) in (3.87) we find

\[ \frac{\sin \varphi}{2} \left( \frac{m_a - m_{0a}}{m_{0a}} \right) = \pm K_e \]

By taking the plus sign one gets the locking range on the upper side of the centre frequency as

\[ \Omega_u = (m_u - m_{0u})_{\text{upper}} = \sqrt{m_{0u}^2 + K_e^2} + K_e - m_{0u} \]  

(3.90)

The negative sign gives the lower side lock range as

\[ \Omega_l = (m_u - m_{0u})_{\text{lower}} = m_u + K_e - \sqrt{m_{0u}^2 + K_e^2} \]  

(3.91)

From (3.90) and (3.91) one finds that the upper side lock range is greater than the lower side lock range. This comes in because of the asymmetric frequency response characteristics of the tuned circuit.

Looking time \( T_L \) on the other hand, can be easily calculated from (3.63) and this is given by [9, 10],

\[ T_L = \frac{\int \varphi \frac{E}{A(t) \sin \varphi} \, d\varphi}{\varphi_{\text{final}} - \varphi_{\text{initial}}} = \frac{E}{2Q A(t) \sin \varphi} \]  

(3.92)

where \( \varphi_{\text{initial}} \) is the initial value and \( \varphi_{\text{final}} \) indicates the final value which is obviously arc \( \sin \left[ \frac{m_u - m_{0u}}{m_{0u}} \right] \), \( A \) is the final value of \( A \).
But if this value of \( \eta \) is taken, \( T_L \) becomes infinite and \( 1 - X \) usually taken as 90 per cent of the steady state value, in order to have a realistic value of \( T_L \). Thus

\[
\eta_L = 6.9 \text{ arc sin } \left( \frac{a_1}{2} \left( \frac{a_2 - c_1}{a_1 - c_1} \right) \frac{E}{A_c} \right)
\]  
(3.93)

Note that during the time that \( \eta \) goes from \( \eta_L \) to \( \eta_L + 1 \), it will also change. To begin with we assume that \( 1 - X \) does not vary much, and nearly remains at \( A_c \). Thus

\[
T_L = \frac{1}{k \sqrt{1 - X^2}} \ln \left[ \frac{-X \tan \left( \frac{\eta_L}{2} \right) + 1 - \sqrt{1 - X^2}}{X \tan \left( \frac{\eta_L}{2} \right) + 1 + \sqrt{1 - X^2}} \right] \quad \text{ (3.94)}
\]

where

\[
k = \frac{X}{\sqrt{1 - X^2}} \text{ and } X = \frac{a_1}{a_2} \left( \frac{a_2 - c_1}{a_1 - c_1} \right) \frac{E}{A_c}
\]

The variation of \( T_L \) with normalized detuning is shown in Fig. 3.5. It is interesting to study the variation of instantaneous best frequency [5] during lock-in. This can be seen from (3.65).

Fig. 3.5. Variation of the locking time with the detuning of the oscillator with respect to the synchronization signal.
\[ \frac{d \theta}{d K(t)} = \frac{4 \sqrt{1 - K^2}}{(1 - K^2) \exp (-\sqrt{1 - K^2}) + [(1 - \sqrt{1 - K^2}) - (1 + \sqrt{1 - K^2}) \exp (-\sqrt{1 - K^2})]} \] (2.96)

The variation of \( d\theta/dK \) with \( \tau \) is shown in Fig. 3.6. From this curve one may find the transient period of frequency entrainment. Thus

![Diagram](image)

**Fig. 3.6.** Variation of the beat frequency with time of an injection synchronized oscillator.
far, the locking time \( T_L \) has been shown to be a function of the damping ratio \( \zeta \) of the oscillator. Consequently, in order to be able to find the locking time accurately one must couple (3.92) with (3.69)

\[
\frac{dA}{d\theta} = \frac{a_0}{2Q} \left( a_0 \sin \theta - \frac{3}{2} a_0 \cos \theta \right) \left( 1 - \zeta a \right) + \frac{a_0}{Q} \cos \varphi
\]

\[
\frac{dA}{d\varphi} = \frac{a_0}{Q} \left( 1 - \zeta a \right) \left( \frac{a_0}{a_0 - a_1} \right) + \frac{a_0}{a_0} \frac{F}{a} \sin \varphi
\]

We rewrite (3.92) and (3.69) as

\[
T_L \frac{a_0}{2Q} = \int \frac{d\theta}{\varphi} \left( \frac{a_0}{a_0 - a_1} \right) \left( 1 - \zeta a \right) - \frac{F}{a} \sin \varphi
\]

and

\[
\frac{dA}{d\varphi} = \left( 1 - \zeta a \right) \left( \frac{a_0}{a_0 - a_1} \right) + \frac{F}{a} \sin \varphi
\]

Comparing (3.96) and (3.97) and using numerical integration the value of \( T_L \frac{a_0}{2Q} \) may be calculated [9, 10]. Note that the initial value of \( \varphi' \) is obviously unity and the final value of \( \varphi' \) is determined from (3.72), which rewritten is

\[
\frac{F}{a} \left( \sin \varphi \right) - (1 - \zeta a) ^ 2 = \frac{a_0}{a_0 - a_1} \left( 1 + \frac{a_0}{a_0} \right)
\]

\[
\frac{d\varphi}{d\theta} = \frac{a_0}{2Q} \left( \frac{a_0}{a_0 - a_1} \right) \left( 1 + \frac{a_0}{a_0} \right) \frac{F}{a} \sin \varphi
\]

Now, during pul-in, the amplitude of the oscillator will also change. If the fluctuation is not vigorous, and the oscillator is not overdriven, one can appreciate that the locking behaviour is very well defined by

\[
\frac{d\varphi}{d\theta} = \frac{a_0}{2Q} \left( \frac{a_0}{a_0 - a_1} \right) \left( 1 + \frac{a_0}{a_0} \right) \frac{F}{a} \sin \varphi
\]
In deriving the above equation, use is made of (3.96) and (3.85c). Thus the locking time is given by

$$\frac{\partial \Theta}{\partial T} = \frac{\cos \gamma}{\sqrt{1 + 0.25 \Delta}} + \cos \frac{\Delta}{\sqrt{1 + 0.25 \Delta}} \sin \gamma$$

(3.98)

where $\gamma$ is the initial value of $\gamma$,

$$\Delta = \frac{\omega_1 (\omega_2 - \omega_0)}{\omega_0 (\omega_2 - \omega_1)}$$

and $\Delta$ is defined as the final value, which is taken to be about 90; of the steady state value of $\Phi_1(\Phi_2)$ given by

$$\Delta + 0.5F \Delta \cos \gamma_1 - F \sin \gamma_1 = 0$$

i.e.

$$\gamma_1 = \arcsin \frac{F}{\sqrt{1 + 0.25 \Delta}} + \arctan \frac{0.5 \Delta}{\Delta}$$

Equation (3.98) is rewritten as

$$\frac{\partial \Theta}{\partial T} = \int \left[ \frac{A + B}{m \sin u} + \frac{C \cos u}{m \sin u} \right] du$$

where

$$u_0 = -\gamma_1 + \arctan \frac{0.5 \Delta}{\Delta}$$

$$u_0 = -\gamma_1 + \arctan \frac{0.5 \Delta}{\Delta}$$

$$A = \frac{0.25 \Delta}{1 + 0.25 \Delta}$$

$$B = \frac{1}{(1 + 0.25 \Delta)^3}$$

$$C = \frac{0.25 \Delta}{1 + 0.25 \Delta}$$

and

$$m = \frac{\Delta}{\sqrt{1 + 0.25 \Delta}}$$

Equation (3.98a) can be easily integrated as

$$\frac{\partial \Theta}{\partial T} = -A(u - u_0) + \frac{B}{\sqrt{1 - m^2}} \ln(Z/e)$$

$$- C \ln \left( \frac{m + \sin u}{m + \sin u} \right)$$

(3.98b)
where
\[
Z_f = \frac{1 + \sin \omega T + \sqrt{1 - m^2 \cos \omega T}}{m + \sin \omega T}
\]
and
\[
Z_i = \frac{1 + \sin \omega T + \sqrt{1 - m^2 \cos \omega T}}{m + \sin \omega T}
\]

Variation of the locking time with the detuning is shown in Fig. 3.5.

3.4 Spectral Character of Unlocked Driven Oscillator [13, 14, 15]

Let us simplify the situation by assuming that the strength of the synchronizing signal is small compared to the oscillator amplitude. Hence amplitude modulation is negligibly small which is almost true in most of the practical situations. Thus we write the oscillator output [10] as
\[
v_f(t) = A_o \cos (\omega_f t - \varphi) = A_0 \cos \omega_f t \cos \varphi + A_0 \sin \omega_f t \sin \varphi
\]  
(3.99)

Remembering that
\[
sin \varphi = Y = -\frac{1}{K} \frac{d \varphi}{d t}
\]

one finds from (3.15)
\[
\sin \varphi = r - 2 \sqrt{X^2 - 1} \frac{1}{2} (1 - 1)^{\pi} \cos \left[2(\beta - \beta_0)\right]
\]  
(3.100)

Now using the identity
\[
\cos \varphi = \frac{1 - \tan^2 \varphi/2}{1 + \tan^2 \varphi/2}
\]  
(3.101)

(3.29) and (3.27), one can rewrite (3.101) as
\[
\cos \varphi = \frac{X^2 - (1 + \tan \beta \tan \beta_0)^2}{X^2 + (1 + \tan \beta \tan \beta_0)^2}
\]  
(3.102)

Noting that
\[
\cos \beta_0 = \frac{1}{X} (\text{ref. 3.27})
\]
we get from (3.102)
\[
\cos \varphi = \frac{\cos \beta - \cos \beta}{\cos \beta + \cos \beta}
\]

i.e.,

\[
\cos \varphi = -\frac{\sqrt{x^2 - 1} \sin (2\beta - \beta)}{x + \cos (2\beta - \beta)}
\]

using the identity

\[
L_0 \left[ \frac{1}{1 + r^2} \right] = -\frac{r}{1 + r^2} + \frac{x}{x + \cos \theta}
\]

which on putting the value of \(r\) reduces to

\[
L_0 \left[ \frac{1}{1 + r^2} \right] = -\frac{x}{x + \cos \theta}
\]

Hence from (3.32) and (3.104) we find

\[
\frac{\sin \theta}{x + \cos \theta} = -\frac{2}{x} \left( -1 \right)^{\nu \eta} \sin (2\beta - \beta_0)
\]

Thus from (3.104) and (3.105), one gets

\[
\cos \varphi = 2 \sqrt{x^2 - 1} \sum_{\nu \eta} \left( -1 \right)^{\nu \eta} \sin (2\beta - \beta_0)
\]

From (3.99a), (3.100) and (3.106), one gets

\[r(t) = \frac{A_0 \sin \omega t + 2A_0 \sqrt{x^2 - 1} \left( -1 \right)^{\nu \eta} \sin \left[ \omega t + \frac{\pi}{4} \right] - K_0 \sqrt{x^2 - 1} \eta \beta_0 - \eta K_0 \sqrt{x^2 - 1}}{x^2 - 1}
\]

where

\[
v_0 = \Omega + \omega
\]

For the sake of understanding the meaning of (3.107) let us consider the following cases:

Case I: Strength of the Synchronising Input is Small Compared to that of the Oscillator:

In this case, obviously \(x\) is large so that one can write

\[
r \approx \frac{1}{2x^2}, \quad \sqrt{x^2 - 1} \approx \frac{1}{4}
\]

and

\[
r \sqrt{x^2 - 1} \approx \frac{1}{4}
\]
Thus the output \( r(t) \) in this case can be approximately represented as

\[
r(t) = A_0 \left( \frac{K}{2\pi} \right) \sin (\omega_0 + \Omega) t - A_0 \sin (\omega_0 t + \delta_0) + A_0 \left( \frac{K}{2\pi} \right) \sin ((\omega_0 - \Omega)t + \delta_0)
\]

where

\[
\delta_0 = \phi_0 - K \sqrt{\Omega^2 - 1} t
\]

\[
\delta_0 = 2\phi_0 - K \sqrt{\Omega^2 - 1} t
\]

Thus in this case the output of the oscillator consists of three components; one at the free-running frequency of the oscillator, and two other components that are symmetrically situated around \( \omega_0 \) [1].

This represents an almost symmetrical spectral characteristic of the unlocked driven oscillator. Sideband components are of much smaller strength than the central one.

**Class II: Strength of the Synchronizing Input is Appreciable but not large:**

So that \( \Omega \) is close to \( K \), i.e., \( \tau \approx \Omega/K \)

Therefore, from Eq. (3.107) one finds

\[
r(t) = A_0 \left( \frac{\Omega}{K} \right) \sin \omega_0 t - 2A_0 \sqrt{\Omega^2 - 1} \left( \frac{\Omega}{K} \right) \sin [(\omega_0 - K \sqrt{\Omega^2 - 1})t + \delta_0]
\]

\[
+ 2A_0 \sqrt{\Omega^2 - 1} (\Omega/K^2) \sin [(\omega_0 - 2K \sqrt{\Omega^2 - 1})t + \delta_0]
\]

This indicates that there are no spectral components above \( \omega_0 \) but a large number of spectral components with progressively decreasing strength is present below \( \omega_0 \). The spectral distribution is quite asymmetric.

### 3.5 Remarks

This chapter begins with an introduction to the phenomenon of injection synchronization in class \( A_1 \) oscillators in simple physical terms based upon the concept of in-phase and quadrature components of the gain parameter of the active device. After this, usually reviews the materials which are available in textbooks but it brings out certain salient features, hitherto unsuspected. This is with respect to asymmetric properties of the injection synchronized oscillator.
such as Locking Range, Locking Time, Jump and Hysteresis, cropping up due to the asymptotic nature of the frequency response characteristic of the tuning element of the oscillator. Finally, this chapter concludes with certain remarks regarding the unlocked behaviour of an oscillator in response to a forcing signal. This explains the non-asymptotic spectral character of an unlocked driven oscillator.

It is quite interesting to note that though the asymptotic properties of the locked oscillators is quite pronounced near the centre frequency of the oscillators, yet it vanishes when one moves away from the centre frequency. The present theory does not indicate this. But if one takes into account higher order terms, this can be shown.

REFERENCES


CHAPTER 4
RESPONSE TO NOISE-FREE MODULATED SIGNALS

4.1 Introduction

Injection locked oscillators find many uses in practice. For example, they can be used to reduce the depth of modulation of an AM signal [1], they can be used as AM/PM converters [2,3], they are employed as amplifiers for angle modulated signals (4, 5, 6), etc. In this chapter, we will present analytical techniques for studying the response characteristics of an injection locked oscillator to amplitude and frequency modulated signals.

4.2 Response to an AM Signal

Let us assume that the synchronizing input and the output of the oscillator are respectively given by

\[ \eta = E(1 + m \sin \omega_m t) \sin \omega_f \]

and

\[ \nu = A(t) \sin (\omega_f t + \phi) \]

(4.1) and

(4.2)

Therefore, following the procedures of Chapter 3, it can be shown that the amplitude and phase equation of the oscillator are given by

\[ \frac{dA}{dt} = \frac{\omega_m}{2Q} \left[ C_1 - \frac{3}{4} C_4 A^2 \right] A + \frac{\omega_m}{2Q} E(1 + m \sin \omega_m t) \cos \varphi \]

(4.3)

and

\[ \frac{d\varphi}{dt} = \Omega - \frac{\omega_m}{2Q} \frac{E(1 + m \sin \omega_m t)}{A(t)} \sin \varphi \]

(4.4)

where

\[ \varphi = (\omega_1 - \omega_2) t - \theta(t) \]
In going through the above equations one appreciates that it is very difficult to solve the problem for any value of the modulating index and arbitrary value of the modulating frequency in relation to synchronization range. In the following we will illustrate the case, when the modulating frequency is small compared to the looking range. For the other case when the modulating frequency is large, the reader is advised to refer to the work [10].

4.3 Modulating Frequency Smaller than the Synchronization Range (Quasi-stationary Approximation)

Here we assume that the modulating frequency is small compared to the synchronization range without imposing any restrictions on the modulating index. In this case it is not hard to see that the time constant of phase synchronization is small compared to the modulating period, and as such it may be assumed that sufficient time has been given to allow the phase and amplitude of the oscillator to attain virtually steady state values during each part of the modulating period. Thus the response to an amplitude modulated signal can be studied by considering the steady state equations.

Putting

\[
E_0 = \frac{E(1 + \mu \sin \omega_0 t)}{\omega_0 A_0}, \quad \Omega = \frac{\omega}{\omega_0}, \quad \alpha = \frac{\Omega}{\omega_0} \text{ and } \Omega = \frac{\Omega}{\omega_0}
\]

One can rewrite (4.3) and (4.4) as

\[
\frac{dx}{dt} = \frac{\omega_0}{2\mu}(1 - \alpha^2) a + \frac{\omega_0}{2\mu} E_0 \cos \varphi \tag{4.5}
\]

and

\[
\frac{dy}{dt} = \Omega - \frac{\omega_0}{2\mu} E_0 \sin \varphi \tag{4.6}
\]

Assuming quasi-stationary operation, one has from (4.5) and (4.6)

\[
(1 - \alpha^2) \varphi_x + \left(\frac{\omega_0}{2\mu}\right) \varphi_x = E_x^0 \tag{4.7}
\]

or

\[
(1 - \alpha^2) \varphi_x + \mu \varphi_x = E_x^0 \tag{4.7a}
\]
where

\[ p = 2q \left( \frac{\Omega}{a_0} \right) \]  

(4.7b)

Moreover, from (4.7a) one gets

\[ a \sqrt{1 - a^2} + \frac{\rho}{a^2} = E \]  

(4.7c)

Note that \( E \) consists of an unmodulated part \( E_0 \) and a modulated part. Corresponding to the unmodulated part, let us assume a value \( a_0 \) for \( a \). Thus putting \( a = a_0 + x \) and \( E_1 = E_0 + y \)

\[ y = f'(a_0) x + \frac{f''(a_0)}{2} x^2 + \ldots \quad 0 < x < 1 \]  

(4.8)

where

\[ f(a_0) = \alpha_0 \sqrt{1 - a^2} + \frac{\rho}{a^2} = E_0 \]  

(4.9)

From (4.8) one can write

\[ x = a y + a_0 y^2 + a_0^2 y^3 + \ldots \]  

(4.10)

which on comparison with (4.8) yields

\[ \frac{1}{a_0} - f'(a_0) = \frac{(1 - a_0^2)(1 - 3a_0^2) + \rho^2}{(1 - a_0^2) + \rho^2} \]  

(4.11)

\[ a_0 = -\frac{f''(a_0)}{f'(a_0)} \]  

(4.12)

\[ a_0 = \frac{\alpha_0}{f''(a_0)} \]  

Then one gets

\[ a = a_0 + a_0 y + a_0^2 y^2 + \ldots \]  

(4.13)

i.e.,

\[ a = a_0 \left( 1 + \frac{a_0}{a_0} y + \frac{a_0^2}{a_0} y^2 + \ldots \right) \]  

(4.14)

Note that

\[ E_0 = \frac{E}{C_y} a_0 \left( 1 + \sin \omega_{\text{sat}} \right) \]  

\[ E_0 = \frac{E}{C_y} a_0 \]  

i.e.,

\[ y = E_0 \sin \omega_{\text{sat}} \]
\[ a = a_0 \left[ 1 + \frac{mE_0}{2\alpha_0} \left( \frac{2}{\alpha_0} \ldots \right) + \frac{mE_0}{4\alpha_0} \left( \frac{3}{\alpha_0} \ldots \right) + \ldots \right] \]
\[ \sin \omega_m t + \frac{mE_0}{2\alpha_0} \frac{2}{\alpha_0} \sin \omega_m t \cos 2\omega_m t + \ldots \] (4.15)

To appreciate the implication of (4.15), let us consider the underdriven case, i.e., \(E_0 \ll 1\). Thus,
\[ a = a_0 \left[ 1 + \frac{mE_0^2}{2\alpha_0} \right] + mE_0 a_0 \frac{2}{\alpha_0} \sin \omega_m t \cos 2\omega_m t \] (4.15a)

Thus the amplitude of the synchronized oscillator is modulated not only at the frequency of the modulating signal but also at the twice frequency of the modulating signal. Thus the effective amplitude modulation index of the oscillator output (cf. 4.15a) and (4.11)
\[ m_{eff} = \frac{mE_0}{\alpha_0} \sqrt{\left( \frac{1}{\alpha_0^2} \right) \left( \frac{1-\beta a}{1-3\beta a} \right) + \rho^2} \]

i.e., from (4.9),
\[ m_{eff} = m \sqrt{\frac{\left( 1-a \right)^2 + \beta^2}{\left( 1-3a \right) + \beta^2}} \] (4.16)
The value \(\alpha_0\) is to be computed from (4.9). From (4.16) it is seen that the output amplitude modulation index for the in-line case (i.e., \(\beta = 0\)) becomes smaller than that due to the off-tuned carrier. When the strength of the carrier is small compared to that of the free-running oscillator, \(\alpha_0\) is nearly unity. As such the output of the oscillator will be almost modulation free (cf. 4.11 and 4.15). Not only this is true but also \(m_{eff}\) is much less than \(m\), the input modulation index of the incoming carrier. The variation of \(m_{eff}\) is shown in Fig. 4.1. Moreover, note that when the input modulation index is small, the output of the oscillator in amplitude modulated with almost no distortion. Thus from (4.15) we have
\[ a = a_0 \left[ 1 + \frac{mE_0 a_0}{2\alpha_0} \frac{\sin \omega_m t}{\cos 2\omega_m t} \right] \] (4.17)
Let us now consider the phase modulation of the oscillator.
In the quasi-steady state operation, the instantaneous phase shift or modulation of the oscillator is (cf. 4.66) given by
\[
\sin \varphi = \frac{\Omega}{a} \frac{2q E_0}{1 + m \cos \omega_M t} \quad (4.18)
\]

Substituting for \(a'\) from (4.17), we get
\[
\sin \varphi = \frac{\Omega}{a} \left( \frac{2q E_0}{1 + m \sin \omega_M t} \right) \quad (4.18a)
\]

where
\[
K = \frac{a}{a_0} = \frac{E^2}{Q^2} = \frac{a_0}{a_0} E
\]

denotes the synchronization band of the oscillator in the absence of modulation. Thus putting
\[
r = \frac{1 - \sqrt{1 - m^2}}{m}
\]

one can easily show
\[
\sin \varphi = \frac{\Omega}{K \sqrt{1 - m^2}} \left[ 1 + \left( \frac{mE_0}{a_0} - 2r \right) \sin \omega_M t + \ldots \right] \quad (4.19)
\]

From (4.19) it is observed that the phase of the oscillator will be modulated but the phase modulation will be faithful only when \(m\) is small, in which case one has
\[
\sin \varphi = \frac{\Omega}{K \sqrt{1 - m^2}} \left[ 1 + m \left( \frac{E_0}{a_0} - 1 \right) \sin \omega_M t \right] \quad (4.20)
\]
Thus, in order to keep the distortion small, one has to choose a small value of $Q/K$. Thus

$$\tau_{lo} = \frac{\Omega}{K} \sqrt{1 - \frac{m}{1 + m\left(E_0 - \frac{1}{2g}\right) \sin \omega_{in}}}. \quad (4.21)$$

Thus it is seen that an ISO can be used as an AM-PM converter [5, 10] for small values of input modulation index, provided the oscillator is synchronized with a small demodulation.

4.4 Response to an FM Signal

We have already seen that in the synchronized condition, an oscillator follows the instantaneous frequency of the synchronizing signal. In this section we will examine the extent of phase following behaviour of the synchronized oscillator with respect to its angle modulated signal $E \cos (\omega_t + \Phi(f))$. Assuming the oscillator output as $v_o = A \cos (\omega_0 t + \Phi)$ and $\Phi = \Phi - \Phi_0$, one can easily show that the system equations are given by

$$\frac{dv}{dt} = -\frac{e_0}{2q} (1 - \alpha) A - \frac{e_0}{2q} F \cos \Phi \quad \text{and} \quad (4.22)$$

$$\frac{d\Phi}{dt} = \frac{\Omega}{2q} \sin \Phi + \frac{dE}{dt} \quad \text{where} \quad F = \frac{E}{C \frac{dv}{dt}} + Q/C, \quad \Phi = \Omega - \theta_0. \quad (4.23)$$

The solutions of the above equations cannot be had for arbitrary values of $dE/dt$. On the contrary, useful results can be had by considering particular situations [9, 12, 13] depending upon the nature of $dE/dt$. Thus in the following subsections, the oscillator behaviour will be analysed depending upon the relation between the modulating frequency and the locking range.

4.4.1 Modulating Frequency Large Compared to the Locking Range (Monotone Modulators)

Let us assume that $\Phi = \sin \omega_{in}$ and $\omega_m > \frac{E}{2q} A_0^2$ where $A_0$.
Phase Lock Theories and Applications

corresponds to the normalized amplitude of the oscillator, when the modulation is absent. Remembering

\[ \eta = E \cos (\omega t + \varphi) \]

it is easily shown that

\[ \eta(t) = E J_0(\mu) \cos \omega_0 t + E J_1(\mu) \cos (\omega_0 + 2\omega_0 \eta) t + \ldots \]

Here \( J_0(\mu) \) is the Bessel's function of the first kind and \( \mu \) th order with the argument \( \mu \). One easily finds that the sideband components of the incoming signal will be outside the locking band; and the oscillator behavior, on the average, will be governed by the carrier component \( E J_0(\mu) \cos \omega_0 t \). Thus the performance of the oscillator, on the average, will be given by

\[ \begin{bmatrix} \frac{d\eta}{dt} \end{bmatrix} = \begin{bmatrix} \Omega - \frac{\mu^2}{4} J_0(\mu) \end{bmatrix} \begin{bmatrix} E \sin \varphi \end{bmatrix} \]  

(4.24)

In writing (4.24) it has been tacitly assumed that the strength of the synchronizing signal is small compared to that of the oscillator. As such we do not take into consideration the amplitude equation. Therefore, the maximum locking range is given by

\[ \Omega \approx \frac{\mu_0^2}{4} J_0(\mu) \frac{F}{A_0} = \frac{\mu_0}{2} \frac{E}{J_0(\mu)} \]  

(4.25)

where \( \mu_0 \) has been assumed to be nearly equal to unity, because the strength of the carrier is small. This gives an approximate locking range. A more accurate expression for the locking range can be had by first assuming that the system is operating under locked condition with certain phase error \( \varphi_0 \) signifying the initial difference of frequencies between two oscillations with respect to the maximum synchronization range of the oscillator. Thus one assumes

\[ \varphi = \varphi_0 + \frac{M_0}{2} \sin (\varphi_0 \sin (\mu_0 t + \varphi_0) \]  

(4.26)

Moreover, in most of the practical situations one can ignore the amplitude fluctuation. Thus (4.23) approximates to

\[ \frac{d\eta}{dt} = \Omega - K \sin \varphi + \frac{d\varphi}{dt} \]  

(4.27)
where

$$K = \frac{\eta_0}{2d} + \frac{\eta_0}{2d} + \frac{E}{2Q} A_0$$

(4.28)

The oscillator is said to be locked to the FM signal provided the average value of the instantaneous frequency error is zero. That is,

$$\left< \frac{d\phi}{dt} \right> = 0 = -K \left< \sin \varphi \right>$$

(4.29)

Therefore, from (4.26) and (4.29),

$$\Omega = J(M)K \sin \varphi$$

(4.30)

Moreover, for locking to occur, \( \varphi \) should satisfy

$$\varphi + M_4 = \pi/2$$

(4.31)

i.e.,

$$\Omega = K(M_4) \cos M_4$$

(4.32)

The value of \( M_4 \) can be found by substituting the assumed solution of \( \varphi \) in (4.23) and applying the method of harmonic balance [11]. Thus for the fundamental component:

$$\omega_0 M_4 \cos (\omega_0 t + \phi) = -2K \cos \varphi_0 J(M) \sin (\omega_0 t + \delta)$$

Equating the coefficients of \( \sin (\omega_0 t + \delta) \) and \( \cos (\omega_0 t + \delta) \) one gets

$$\omega_0 M_4 = \omega_0 \cos \delta$$

(4.33)

$$2K(M) \cos \varphi_0 = \omega_0 \sin \delta$$

Therefore,

$$M_4 = \left[ \omega_0^2 + K^2 \left( \frac{J(M)}{\omega_0} \right)^2 \right]^{1/2} \cos \varphi_0$$

(4.34)

Noting that \( \Delta \), the maximum frequency deviation is equal to \( \omega_0 \delta \) and referring to (4.31) one has

$$M_4 = \left( \frac{\omega_0 \delta}{K} \right)$$

(4.35)

Note that if \( \omega_0 \delta \) is large compared to \( K \), the expression (4.32) for the locking range converges to that given by (4.25).
4.4.2. Response to an FM Signal Having Modulating Frequency Small Compared to the Locking Range

This section deals with the study of the response of an ISO to an FM signal having modulating frequency small in comparison with the locking range. If \( \frac{\Delta f}{f_0} \) is small compared to \( \omega_0 \frac{E}{2Qf_0} \), it is not hard to see that the locking time of the oscillator will be small in comparison with the period of the modulating signal \( \omega_0 \) \( \text{mHz} \). Thus, during the process of lock-in, \( \frac{\Delta f}{f_0} \) may be treated as constant and the instantaneous frequency error \( \left( \Omega + \frac{df}{dt} \right) \) may be considered to be independent of time. Therefore, the locking range \( \Omega \) of the injection-synchronized oscillator in this case is to be found from

\[
\text{(4.33)}
\]

where \( p(\Delta) \) denotes the probability density function of \( \Delta \), given by

\[
\text{(4.36)}
\]

and for this case it is not difficult to show that

\[
\text{(4.37)}
\]

From (4.35) through (4.37) one can easily show that

\[
\Omega = K \left( 1 - \frac{2m_{\text{min}}}{nK} \right)
\]

where

\[
K = \omega_0 \frac{E}{2Qf_0}
\]

The relation (4.38) gives the variation of the locking range with the modulating frequency and modulation index when the modulating frequency is small.

4.4.3. Filtering Property

To study the filtering property, let us assume that the center frequency of the ISO is equal to that of the carrier. This is assumed because in this case the oscillator has the highest capability of tracking. Rivera [9, '2] the phase equation as
\[ \frac{dy}{dt} = -K \sin \varphi + \frac{d\theta}{dt} \]

i.e.,

\[ \frac{dy}{dt} = -K \sin \varphi + \Delta \cos \omega t \]

Put

\[ \omega t = \lambda, D = \frac{\Delta}{K}, F = \frac{\sin \lambda}{K} \]

then

\[ \frac{dy}{dx} = -\frac{1}{F} \sin \varphi + \frac{D}{F} \cos \lambda \]

(4.39)

To start with, one assumes that \( \varphi = \varphi_0 \), and writes in the first approximation (\( \phi \approx \phi_{0m} \))

\[ \frac{d\phi_{0m}}{dx} + \frac{\phi_{0m}}{F} = \frac{D}{F} \cos x \]

(4.40)

That is,

\[ \phi_{0m} = \frac{D/F}{1 + \left( \frac{1}{F} \right)} \left( \cos x \right) + 
\]

(4.41)

Now assume that the second order solution is of the form

\[ \phi_m = \phi_{0m} + \eta \]

(4.42)

where \( \eta \) is a small quantity. From (4.41) and (4.42) we get

\[ \frac{d\phi_m}{dx} = \frac{d\phi_{0m}}{dx} + \frac{d\eta}{dx} \]

Putting \( \frac{d\phi_m}{dx} \) from (4.37) one finds that

\[ \frac{d\phi_m}{dx} = + \frac{D/F}{1 + \left( \frac{1}{F} \right)} \left( \cos x - \sin x \right) \]

(4.43)

Again utilizing (4.39) i.e.,

\[ \frac{d\phi_m}{dx} = -\frac{1}{F} \sin \phi_m + \frac{D}{F} \cos \lambda \]

(4.39a)

and expanding \( \sin \phi_m \) as

\[ \sin \phi_m = \sin (\phi_{0m} + \eta) \]

\[ \sin \phi_m \approx \sin \phi_{0m} + \cos \phi_{0m} \eta \]
\[
\begin{align*}
&= \left[ 2i \left( \frac{DlF}{1 + \frac{F}{Z}} \right) l \left( \frac{DlF}{1 + \frac{F}{Z}} \right) \cos x \\
&+ 2i \left( \frac{DlF}{1 + \frac{F}{Z}} \right) l \left( \frac{DlF}{1 + \frac{F}{Z}} \right) \sin x \\
&+ \psi \left( \frac{DlF}{1 + \frac{F}{Z}} \right) l \left( \frac{DlF}{1 + \frac{F}{Z}} \right) \right] \\
\end{align*}
\]

(4.39b)

We get from (4.39a), (4.39b) and (4.39)

\[
\frac{ds}{dx} + R \eta = Y \cos \alpha + Z \sin x
\]

(4.44)

Thus,

\[
\psi \equiv Mq \cos \omega t + m \sin \omega t
\]

(4.45)

where

\[
\begin{align*}
m_0 &= \frac{m}{1 + \frac{1}{F \alpha}} \left( \frac{Y}{F \alpha} \right)
\end{align*}
\]

(4.46)

\[
\begin{align*}
m_0 &= \frac{m F}{1 + \frac{1}{F \alpha} + \frac{F Y}{1 + \frac{Z}{F \alpha}} - \frac{Z}{F \alpha}}
\end{align*}
\]

(4.47)

\[
\begin{align*}
P &= \frac{1}{F \alpha} l \left( \frac{m}{1 + \frac{1}{F \alpha}} \right) \left( \frac{m F}{1 + \frac{1}{F \alpha}} \right)
\end{align*}
\]

(4.48)

\[
\begin{align*}
y &= m - \frac{m}{1 + \frac{1}{F \alpha}} \left( \frac{m}{1 + \frac{1}{F \alpha}} \right) l \left( \frac{m F}{1 + \frac{1}{F \alpha}} \right)
\end{align*}
\]

(4.49)

\[
\begin{align*}
\varepsilon &= \frac{1}{F \alpha} l \left( \frac{m}{1 + \frac{1}{F \alpha}} \right) \left( \frac{m F}{1 + \frac{1}{F \alpha}} \right) + \frac{m F}{1 + \frac{1}{F \alpha}}
\end{align*}
\]

(4.50)

Thus the output of the oscillator can be written as

\[
v_0 = A \cos (\omega t + \phi (t))
\]

(4.51)

where

\[
\phi (t) = (n - m) \sin \omega t - m \cos \omega t
\]

(4.52)

Therefore, the output carrier and sideband powers are respectively given by

\[
(P_0)_{m} = A^2 (\frac{1}{2} \frac{L}{l} (m - m) \eta \frac{\omega}{\psi} (m_0)
\]

(4.53)

\[
(P_{\omega m}) = \frac{1}{4} \frac{2 L}{l} (m - m) J_0^2 (m_0)
\]

(4.54)

The sideband attenuation is defined as
\[ a = \frac{(R)_{\text{out}}}{(R)_{\text{in}}} \]  
\[ (R)_{\text{in}} = \frac{(P)_{\text{in}}}{(P)_{\text{out}}} \]  
\[ (R)_{\text{out}} = \frac{(P)_{\text{in}}}{(P)_{\text{out}}} \quad \text{and} \quad (R)_{\text{in}} = \frac{1 - J_1^2(m)}{J_2^2(m)} \]

Now for the case when the modulating frequency is large compared to the locking range, i.e., \( F \gg 1 \), the sideband attenuation reduces to

\[ a = \frac{J_1^2(m)}{1 - J_1^2(m)} \left( \frac{2}{F} \int \frac{J_1(F)}{J_1(F)} \right) \]

Moreover, for large values of \( F \), the output of the oscillator is

\[ v_o \approx A_0 \cos \left[ \omega_m t + \frac{2}{F} \int \frac{J_1(F)}{J_1(F)} \sin \omega_m t \right] \]

Therefore, the ratio of the first sideband power to the carrier power is given by \( P_1 = \frac{J_1(F)}{J_1(F)} \)

\[ P_1 = \left[ \frac{2}{F} \int \frac{J_1(F)}{J_1(F)} \right] = \frac{K}{\omega_m} J_1(m) \]

At this point let us note the corresponding value of the ratio when the incoming FM signal is passed through a single-tuned circuit of quality factor \( Q \). This is given by

\[ R_q = \frac{1}{1 + Q \left( \frac{\omega_m}{\omega_0} \right)^2} J_1(m) \]

i.e.,

\[ R_q = \frac{1}{1 + Q \left( \frac{\omega_m}{\omega_0} \right)^2} J_1(m) \]

Therefore, the effective \( Q \) of the oscillator is
\[
Q_{m} = \frac{1}{2} J_{1} (\frac{\omega_{m}}{K})
\]

i.e.,

\[
Q_{m} = \frac{Q}{\omega_{m}(m) \frac{A}{E}}
\] (4.62)

This indicates that the quality factor of the tank circuit is magnified by the ratio of the oscillator amplitude to that of the synchronizing input.

4.4.4. Amplifying Property

From the operation of synchronized oscillator it is seen that when the oscillator is tracking the input FM signal, the output phase variation can be identical with that of the input. If this is achieved, the output of the oscillator will be an exact replica of the input but with a large amplitude compared to that of the synchronizing input. This gives the amplifying property of the IFO. Referring to (4.32) and (4.33) it is seen that for an in-tunedcarrier the error \('M'\) will be small provided \(K\) is large compared to \(\omega_{m}\). One more condition is required that the locking range should be adequate so that oscillator does not slip out of synchronization. Finally, the distortion should be low. Recall that

\[
\frac{d\phi}{dt} = \Omega - K \sin \phi + \frac{d\phi}{dt}
\] (4.63)

i.e.,

\[
\frac{d\phi}{dt} = \Omega - K \sin \phi + \frac{A}{K} \cdot K \cos \omega_{m} t
\] (4.63a)

Changing the variable 't' to \(z = Kt\) one gets

\[
\frac{d\phi}{d\psi} = X - \sin \psi + mF \cos Fy
\] (4.64)

where

\[
X = \frac{\Omega}{K}, \quad y = \frac{\omega_{m}}{K}
\]

To solve (4.64), let us first replace \(\sin \psi\) by \(\psi\), and write the steady state first order solution as

\[
\psi = \psi' + \frac{mF}{\sqrt{1 + \tan^{2} F}} \cos (Fy - \tan^{-1} F)
\] (4.65)
(4.65) It is seen that even when $v$ is small, the distortionless solution is obtained when $F$ is small, i.e., one onde lock range (5) is large compared to the modulating frequency $(\omega_m)$. Therefore, under this condition, i.e., $F < 1,$

$$\psi(t) = X + mF \cos \omega_m t$$

(4.65)

To achieve the second order solution, one replaces $\sin \phi$ by $\frac{x}{6}$ and writes (4.64) as

$$\frac{dp}{dt} + \psi_t - \frac{\psi}{2} = X + mF \cos \phi y$$

(4.67)

Therefore, again under the condition that $F$ is much less than unity, one has

$$\psi(t) = X + mF \cos \omega_m t + \frac{1}{6} (X + mF \cos \omega_m)$$

(4.68)

Thus the distortion is seen to become small when $X$ is zero, because second harmonic distortion becomes zero. However, the third harmonic distortion remains. Thus an ISO can act as a faithful amplifier for an angle modulated signal provided the deviation is small. Usually, the modulating signal has a complex character, and in many cases it is approximated by a band-limited Gaussian noise [14, 15] having the power spectral density

$$S_x(u) = \frac{(2\pi)^2}{2f_x - f_0} f_x \ll |u| < f_0$$

(4.69)

$$S_x(u) = \frac{(2\pi)^2}{2f_x^2}, f_x > f_0$$

(4.70)

where $2^3$ is the mean square frequency deviation, $f_x$ and $f_0$ are respectively the highest and the lowest modulating frequencies of the signal. Proceeding exactly in an analogous way, one can easily show that

$$\psi(t) = \frac{\partial \psi}{\partial t}$$

(4.71)
Therefore, the second order solution can be written as
\[
\frac{d^2 \theta(t)}{dt^2} = \frac{\partial \theta}{\partial t} - K \left[ \psi(t) - \frac{\theta}{2} \right]
\]  
(4.72)

That is, when \( K \) is large compared to the highest modulating frequency,
\[
\psi(t) = \frac{\partial \theta}{\partial t} + \frac{1}{\dot{\theta}} \left( \frac{\partial \theta}{\partial t} \right)^* \quad \text{(4.73)}
\]

Therefore, the third order distortion term is
\[
\theta(t) = \frac{1}{6} \left( \frac{\partial \theta}{\partial t} \right)^* \quad \text{(4.74)}
\]

Therefore, the power spectral density of \( \theta(t) \) is
\[
S_{\theta}(\omega) = \left( \frac{1}{2\pi} \right)^2 |S_{\psi}(\omega)|^2 \quad \text{(4.75)}
\]

where \( S_{\psi}(\omega) \) denotes the power spectral density of \( \partial \theta/\partial t \). In order to evaluate \( S_{\theta}(\omega) \), one uses the following relation
\[
S_{\theta}(\omega) = \int_{-\infty}^{\infty} R_{\theta}(t) e^{-i\omega t} dt \quad \text{(4.75)}
\]

where \( R_{\theta}(t) \) is the autocorrelation function of \( \theta \). That is,
\[
R_{\theta}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\theta}(\omega) e^{i\omega t} d\omega \quad \text{(4.77)}
\]
\[
R_{\theta}(\tau) = \frac{(2\pi)^2}{\dot{\theta}^2} \sin \frac{\omega_0 \tau}{2\dot{\theta}} \quad \text{for } \tau > \frac{\omega_0}{\dot{\theta}} \quad \text{(4.78)}
\]

Therefore,
\[
S_{\theta}(\omega) = \int \left( \frac{(2\pi)^2}{\dot{\theta}^2} \sin \frac{\omega_0 \tau}{2\dot{\theta}} \right)^2 (3\dot{\theta}^2 - f^2)^{-1} \quad f \ll \dot{\theta} \quad \text{(4.79)}
\]

Therefore, the third order distortion-to-signal power ratio is given by
\[
\frac{P_3}{\dot{P}_2} = 10 \log \left[ \frac{|S_{\theta}(\omega)|^2}{S_{\theta}(\omega)} \right] + 10 \log \left( \frac{3\dot{\theta}^2}{3\dot{\theta}^2 - f^2} \right) \quad \text{for } f \ll \dot{\theta} \quad \text{(4.80)}
\]
where $B_k = |R_k|$, denotes two-sided lock range.

4.5 Remarks

In this chapter we have considered the response of an oscillator to pure CW amplitude and frequency modulated signals. Simple minded approaches have been developed to illustrate the possibility of using an injection synchronized oscillator as

1. an AM to PM converter,
2. a narrow band filter, and
3. an amplifier for angle modulated signals.

The analyses have been restricted to the case of an under driven oscillator, and as such amplitude variations have been neglected. The amplitude variation, when taken to consideration complicates the methods of analysis to such an extent that the task of numerical solution is to be invoked. The results of incorporation of the amplitude variation remain basically the same as those of the case when amplitude variations were neglected. However certain modifications are expected.

REFERENCES

Phase Lock Theories and Applications


5.1 Introduction

Injection synchronized oscillators have many interesting properties. In this chapter we will discuss its ability of discriminating a signal against unwanted disturbances accompanying the signal (1–4). We will here consider two types of disturbances, (i) intercoupling tone and (ii) additive Gaussian noise. Filtering action of the synchronized oscillator depends on the quality factor of the tank circuit of the oscillator, the carrier-to-noise power at the input and also on the type of non-linearity incorporated within the oscillator. Discussion will be mainly restricted to the case of low level of synchronization. This is mostly true in practice, because the strength of the incoming signal is small compared to that of the free running oscillator.

5.2 Derivation of System Equations

Assuming the synchronizing signal and the interfering tone to be of the forms \( I \cos \omega_0 t \) and \( J \cos \omega_{af} t \) respectively, and referring to Chapter 3, it is easily shown that the response of a van der Pol oscillator to such a combined signal is given by

\[
\frac{d^2 \phi}{dt^2} - \frac{\phi}{\alpha} + \left( C_1 \phi - C_2 \phi^3 \right) + \omega^2 \phi = \frac{\omega_0^2}{\alpha} \sin \omega_0 t + \omega_{af} \sin \omega_{af} t
\]

(5.1)

Note that \( y \) is a fraction which indicates the strength of the interfering tone in comparison with the synchronizing signal. Again taking the solution of (5.1) as

\[
x = A(t) \cos (\omega_0 t + \varphi(t))
\]

(5.2)
and adopting the procedure of Chapter 3 (Section 3.5), it can be easily shown that the amplitude and phase equations of the forced van der Pol oscillator are given by

\[
\frac{da}{dt} = \frac{\omega_0^2}{2Q} \left( C_1 - \frac{3}{4} C_2 \psi \right) a + \frac{\omega_0^2}{2Q} \left( \cos \varphi + \gamma \cos (\Delta \omega t + \psi) \right)
\]

(5.3)

and

\[
\frac{dp}{dt} = \frac{\omega_0}{2} \left( \sin \varphi - \psi \right) - \frac{\omega_0^2}{2Q} \left( \sin \varphi + \gamma \sin (\Delta \omega t + \psi) \right)
\]

(5.4)

where

\[
\varphi = -\psi
\]

(5.5)

Using the notations of (5.58) through (5.61), the amplitude and phase equations can be rewritten as

\[
\frac{da}{dt} = \frac{\omega_0^2}{2Q} \left( 1 - \alpha \right) a + \frac{\omega_0^2}{2Q} \cos \varphi + \gamma \cos (\Delta \omega t + \psi)
\]

(5.7)

and

\[
\frac{dp}{dt} = \Omega - \frac{\omega_0^2}{2Q} \psi \left( \sin \varphi + \gamma \sin (\Delta \omega t + \psi) \right)
\]

(5.8)

where

\[
\Omega = \frac{\omega_0}{2} \left( \sin \varphi - \psi \right)
\]

(5.9)

Since a general solution of (5.7) and (5.8) is not possible, we will simplify the situation by considering two specific cases, viz. (1) weak interference, and (2) strong interference. To analyze the behavior of the ISO in such a situation, we will assume that the oscillator, instead of being exposed simultaneously to both the signal and the interfering tone, is initially locked to the synchronizing signal and then it is influenced by the interfering tone which does not throw the system out of synchronization. In this condition, the instantaneous phase of the oscillator will fluctuate around the static phase error \(\psi\) at a rate depending upon the difference of frequencies between the synchronizing signal and the interfering tone. Similarly, the amplitude of the forced oscillator will also vary around the steady state value, a rate depending on the frequency difference between the two signals.

The static phase error \(\psi\) gives a measure of the initial detuning of the oscillator with respect to the synchronizing signal.
5.3 Weak Interference Lying Away from the Oscillating Centre Frequency

In this case we assume that the strength of the interfering tone is small, and as such we neglect the amplitude fluctuation of the forced oscillator. Thus we consider the phase equation of the oscillator only, \( \ddot{\varphi} + \omega_0^2 \varphi = 0 \), and assume the following solution for \( \varphi \):

\[
\varphi = \varphi_1 + m \sin (\Delta \omega t + \nu)
\]

(5.19)

It is to be noted that by virtue of the assumption of weak interference, \( m \) is small.

Using the relation

\[
\sin \varphi = \sin (\varphi_1 + m \sin (\Delta \omega t + \nu)) = \varphi_1 + m \cos \varphi_1 \sin (\Delta \omega t + \nu)
\]

(5.11)

\[
\sin (\Delta \omega t + \nu) = \sin (\Delta \omega t + \varphi_1 + m \sin (\Delta \omega t + \nu)) = \varphi_1 \cos (\Delta \omega t + \nu) + m \cos \varphi_1 \sin (\Delta \omega t + \nu)
\]

(5.12)

and inserting (5.10), (5.11) and (5.12) in (5.4) one gets the following equations by the method of harmonic balance:

\[
\frac{\Omega}{K} = \varphi_1 + m \varphi_1 \sin \Delta \omega t \cos \varphi_1 \cos (\Delta \omega t + \nu) - m \cos \nu \sin \Delta \omega t \cos \varphi_1 \cos \varphi_1 \cos (\Delta \omega t + \nu)
\]

(5.13)

\[
\frac{m^2 \omega_0^2}{K} = \sin \nu - 2 \varphi_1 \cos \nu \sin \Delta \omega t \cos \varphi_1 \cos \varphi_1 \sin \Delta \omega t + m^2 \sin \nu
\]

(5.14)

\[
\frac{m^2 \omega_0^2}{K} = \sin \nu - 2 \varphi_1 \cos \nu \sin \Delta \omega t \cos \varphi_1 \cos \varphi_1 \sin \Delta \omega t + m^2 \sin \nu
\]

(5.15)

Multiply (5.14) and (5.15) respectively by \( \cos \nu \) and \( \sin \nu \), and use (5.13) to yield

\[
\frac{\Omega}{K} = \varphi_1 + \frac{m^2 \omega_0^2}{K} \varphi_1
\]

(5.16)

Again from (5.14) and (5.15), one gets

\[
\left( \frac{m^2 \omega_0^2}{K} \right) + 4 \varphi_1^2 \cos \varphi_1 = \mathcal{F}(m)
\]

(5.17)

Thus (5.16) can be rewritten as
\[ \Omega = \frac{2}{K} \sin \varphi_1 + \frac{\rho^2 m (\Delta \omega/K) \varphi_2 (m) \varphi_2 (m)}{2 (\rho^2 \Delta \omega/K)^2} \] (5.19)

when \( m \) is small, we approximately write (5.18), by putting \( \varphi_2 (m) \approx 1 \) and \( \varphi_2 (m) = -m \Delta \omega ) \), as

\[ \Omega = \sin \varphi_1 + \frac{\rho^2}{2} \frac{\Delta \omega/K}{\cos \varphi_1 + (\Delta \omega/K)^2} \] (5.19)

Further from (5.17) one finds

\[ m \Delta \omega = \cos \varphi_1 + (\Delta \omega/K)^2 \rho^2 \] (5.20)

which indicates that the phase modulation due to the interfering tone decreases with the increase of \( m \).

Now to calculate the locking range we note that the maximum value of \( \Omega (\Delta \omega) \), i.e., the locking ratio, can be found from (5.19) by differentiating the left-hand side of (5.19) with respect to \( \varphi_1 \) and putting it equal to zero, i.e.,

\[ \cos \varphi_1 + \frac{\rho^2}{2} \frac{\Delta \omega}{K} \frac{2 \cos \varphi_1 \sin \varphi_1}{\cos^2 \varphi_1 + (\Delta \omega/K)^2} = 0 \] (5.21)

From this one finds that \( \varphi_1 \) is nearly equal to \( \pm \pi/2 \), and one gets the maximum value of \( \Omega (\Delta \omega) \), when \( \Delta \omega \) is large. Therefore the locking range is given by (cf. 5.19 and put \( \varphi_1 \approx \pm \pi/2 \))

\[ \Omega = \pm 1 + \frac{\rho^2}{2} \frac{\Delta \omega}{K} \] (5.22)

Now when \( \Delta \omega \gg K \), we can rewrite (5.22) as

\[ \Omega = \pm \frac{1}{K} + \frac{\rho^2}{2} \frac{\Delta \omega}{K} \] (5.23)

The term within the braces of (5.23) indicates either an increment or a decrement in the locking range of the oscillator over that of the interference-free one (i.e., \( K \)). For example, if one observes the upper-side lock range of the oscillator, i.e., with the positive sign, it will be greater than \( K \) when the interfering tone lies on the upper-side and it will be less than \( K \) when the interfering tone lies on the lower-side. Analogous conclusion can be drawn for the lower-side lock range. It is further seen that the increment or decrement in the locking range depends solely on the strength of the interfering tone and its detuning with respect to the free running frequency of the oscillator. Now referring to Chapter 3 one finds that the term,
within the braces of (5.23), viz.,

\[ \Delta \omega = \sqrt{(\Delta \omega_0)^2 - (\Delta \kappa)^2} \]

signifies the shift of the free running frequency of the oscillator when it is beating with the interfering tone alone. From what has been proved we can draw an interesting conclusion [5, 6]. The locking range of an injection synchronized oscillator in the presence of an interfering tone, when it is away from the oscillator, is equal to the interference-free locking range plus an additional term arising due to the shift of the oscillator frequency due to the interference alone. Or equivalently, we may state that the locking range of an oscillator is such a situation can be computed by considering a new oscillator whose free running frequency has been shifted by an amount depending upon the strength and location of the interfering tone.

Interference filtering property in this case can be easily calculated by rewriting the output of the oscillator as (cf. 5.2 and 5.10)

\[ x = A_0 \cos (\omega t - \varphi - m \sin (\omega t + \alpha)) \]

(5.24)

Following the development of Chapter 4 one can calculate the sideband attenuation as

\[ a = \frac{R_{1s}}{R_0} \]

(5.25)

where

\[ R_{1s} = \frac{P_{1s}}{P_{m}} \]

(5.26)

\[ R_0 = \frac{P_0}{P_m} \]

(5.27)

\[ P_{1s} = \text{Sideband power at the output} = \frac{1}{2}(1 - \mu)P_m \]

(5.28)

\[ P_0 = \text{Carrier power at the input} = \frac{1}{2}P_m \]

\[ P_{1} = \text{Power of the interfering tone} = \frac{1}{2}P_m \]

and

\[ P_s = \text{Power of the synchronizing signal} \]

That is,

\[ a = \frac{1 - \mu^2(m)}{\frac{1}{2}P_m + \frac{1}{2}P_m} \]

(5.29)

where \( m \) is given by (5.26).
When the interfering tone lies close to the oscillator, but not within the locking band then it produces phase modulation as well as amplitude perturbation. Therefore, to get an accurate picture one has to solve the amplitude and phase equations simultaneously. But simultaneous consideration of these perturbations makes it difficult for achieving the solution. However, a fairly accurate result in this case can be obtained by carrying out the analysis in two steps (6, 7).

First, the phase-perturbation of the ISO is taken into consideration, assuming the instantaneous amplitude to remain constant at its average value. In the next step, the amplitude modulation is introduced as a correction term.

5.4 Strong Interference

Depending upon the detuning of the interfering tone with respect to the locking range of the ISO, the following two situations may arise:

1. The interfering tone may be outside the synchronization range of the ISO.

2. The interfering tone may be located inside the lockband of the ISO.

In the former case, to start with, we assume that the oscillator is synchronized to the synchronizing signal, and then gradually increase the strength of the interfering tone. As the strength of the interfering tone is increased, it will exert pull on the oscillator, and as a result it will produce phase and amplitude modulations of the oscillator. Thus with the increase of the strength of the interfering tone a condition will come up, when the oscillator will be unlocked from the synchronizing signal and a further increase of the strength of the interfering tone may pull the oscillator to its own frequency. This leads to the situation when the roles of the synchronizing signal and the interfering tone are interchanged. Similar phenomena may also appear in the latter case.

Thus, in either of the cases, if the strength of the interfering tone is sufficient, the oscillator frequency may jump to that of the interfering tone, exhibiting a sort of "frequency jump" phenomenon in an ISO. For details of this study interested readers may refer to the works (6, 7, 8).
5.5 Signal Contaminated with Gaussian Noise

Usually the signal contaminated with additive white Gaussian noise is passed through a narrow band circuit, the bandwidth of which is large compared to that of the oscillator. As such, the net input signal to the oscillator can be written as

$$i(t) = I \cos (\omega_d t + \theta) + I_d(t)$$

(5.29)

where

$$I_d(t) = \sqrt{2} I_d(t) \cos (\omega_d t) + \sqrt{2} I_d(t) \sin (\omega_d t + \theta)$$

(5.29a)

$L_d(t)$ and $I_d(t)$ are narrowband independent Gaussian variables with one-sided spectral density $I_{ds}$, same as that of $I_d(t)$ [cf. Chapter 1].

The response of a van der Pol oscillator to a signal of the form as denoted by (5.29) can be written as (cf. 3.63-3.64):

$$\frac{dx}{dt} = -\frac{a_0}{2} \left( x - C_0 x^3 \right) + a_0 x = \frac{a_0}{2} \left( \sqrt{2} N_d(t) \cos (\omega_d t) + \sqrt{2} N_d(t) \sin (\omega_d t + \theta) \right)$$

(5.30)

where

$$N_d(t) = \frac{M}{E} \left( \frac{L}{CR} \right) \frac{dL}{dt}(t)$$

(5.31)

$$N_d(t) = \frac{M}{E} \left( \frac{L}{CR} \right) \frac{dI}{dt}(t)$$

(5.32)

In deriving the above-equation it has been assumed that $I_d(t)$ and $I_d(t)$ are slowly varying functions of time. Now to derive the amplitude and phase equations of the oscillator, under the infinitesimal of the noisy signal, we assume that the output of the oscillator is of the form

$$x(t) = A(t) \cos (\omega_d t + \varphi(t))$$

(5.33)

Now following the procedure of section 3.3., it can be shown that

$$\frac{dA}{dt} + \frac{q_0}{2Q} (C_1 - \frac{3}{4} C_4) A + \frac{q_0 E}{2Q} \cos \varphi + \frac{q_0}{2Q} N_d(t)$$

(5.34)

and

$$\frac{d\varphi}{dt} = \frac{q_0}{2Q} \left( q_0 - a_0 \right) - \frac{q_0 E}{2Q} A \sin \varphi + \frac{q_0}{2Q} N_d(t)$$

(5.35)
\[ N_d(t) = \sqrt{2N_0(t)} \cos \varphi - \sqrt{2}N_0(t) \sin \varphi \]
\[ N_s(t) = \sqrt{2N_0(t)} \cos \varphi + \sqrt{2}N_0 \sin \varphi \]
\[ \varphi = 0 - \varphi \]

It is to be noted that since \(N_d(t)\) and \(N_s(t)\) are random variables having the same spectral density as that of \(N(t)\):
\[ n(t) = \sqrt{2N_0(t)} \cos (\omega_0 t + \varphi) - \sqrt{2}N_0(t) \sin (\omega_0 t + \varphi) \]

5.5.1 FDP's of the Phase Error and the Amplitude

Before finding the filtering property of the oscillator, i.e., spectral purity of the output waveform, let us first calculate the probability density distribution functions of the phase error and the amplitude of the oscillator.

Referring to (5.34) and (5.35) it is seen that the amplitude process is much faster than the phase process [11]. As such one may assume that long before the phase process attains the steady state, the amplitude has attained a steady state distribution. Thus we assume that the phase distribution can be found from (5.35) by considering the following equation,
\[ \frac{\delta \varphi}{\delta t} = -\frac{\omega_0}{2Q} E \left\langle A \right\rangle \sin \varphi + \frac{\omega_0}{2} N_d(t) \]

where \(\left\langle A \right\rangle\) is the average value of \(A\), and
\[ \Omega = \frac{\omega_0}{2Q} \left( \frac{\omega_0}{\omega} - \frac{\omega_0}{\omega_0} \right) \]

It can be shown that if the bandwidth of the predetection filter is large compared with the bandwidth of the oscillator, \(N_d(t)\) represents the character of an almost white Gaussian noise with zero mean. Moreover, the derivative of \(\varphi\) does not depend on its past history, but depends only on the value of \(\varphi\) at the instant considered. Therefore, \(\varphi\)-process may be identified with a Markov process (cf. Chapter 1 and Chapter 12). Therefore, the probability density function \(p(\varphi, \dot{\varphi})\) at each instant is dictated by the Fokker-Planck-Kolmogorov equation [14, 15].
\[
\frac{\partial \chi(\theta, t)}{\partial t} = -\frac{1}{2\Theta} \left( A_\phi(\theta) \chi(\theta, t) \right) + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \left( A_\psi(\theta) \psi(\theta, t) \right)
\]

(5.40)

where
\[
A_\phi(\theta) = \lim_{\Delta \theta \to 0} \frac{E(\Delta \phi | \theta)}{\Delta \theta}
\]

(5.41)

and
\[
A_\psi(\theta) = \lim_{\Delta \theta \to 0} \frac{E(\Delta \psi | \theta)}{\Delta \theta}
\]

(5.42)

Note that \(E(u|v)\) denotes the conditional mathematical expectation of \(u\) given the value of \(v\). Now put

\[
\beta = \frac{\nu_0}{2\Theta} \langle A \rangle
\]

(5.43)

and rewrite (5.38) as

\[
\frac{dp}{dt} = (\Omega - \beta E \sin \varphi) \Delta t + \beta \int_{-\infty}^{t+\Delta t} N_d(u) \, du
\]

(5.44)

Therefore,

\[
\Delta \varphi = (\Omega - \beta E \sin \varphi) \Delta t + \beta \int_{-\infty}^{t+\Delta t} N_d(u) \, du
\]

(5.45)

Remembering \(N_d(t)\) has zero mean, it is easily seen that

\[
A_\phi(\theta) = \lim_{\Delta \theta \to 0} \frac{E(\Delta \varphi | \theta)}{\Delta \theta} = \Omega - \beta E \sin \varphi
\]

(5.46)

Moreover, from (5.43)

\[
(\Delta \theta)^2 = (\Omega - \beta E \sin \varphi)^2 \Delta t^2
\]

\[
- 2\beta (\Omega - \beta E \sin \varphi) \Delta t \int_{-\infty}^{t+\Delta t} N_d(u) \, du
\]

\[
+ \beta^2 \int_{-\infty}^{t+\Delta t} \int_{-\infty}^{t+\Delta t} N_d(u) N_d(v) \, du \, dv
\]

(5.47)

Since \(E(N_d) = 0\) one gets from (5.47)
Phase Lock Theories and Applications

\[
A_2(\varphi) = \lim_{\nu \to 0} \left[ \left( \frac{11}{2} \right) \int_{-\nu}^{\nu} E[N_2(u)N_2(v)] \, du \right] \tag{5.48}
\]

Since \( N_2(t) \) is a white Gaussian process with spectral density \( N_2/2 \),
\[
E[N_2(u)N_2(v)] = \frac{N_2}{2} \delta(u - v) \tag{5.49}
\]

Hence from (5.48) and (5.49) one has
\[
A_2(\varphi) = \frac{N_2}{2} \tag{5.50}
\]

Hence the FPK equation is
\[
\frac{\partial}{\partial t} \left[p(\varphi, t)\right] = -\frac{\partial}{\partial \varphi} \left[(\Omega - \beta E \sin \varphi) \cdot p(\varphi, t)\right] + \frac{\partial^2}{\partial \varphi^2} \left[p(\varphi, t)\right] \tag{5.51}
\]

The probability density function in the steady state thus (Ch. 15), obeys
\[
0 = -\frac{d}{d\varphi} \left[(\Omega - \beta E \sin \varphi) \cdot \omega(\varphi)\right] + \frac{\beta N_2}{4} \cdot \frac{d^2\omega(\varphi)}{d\varphi^2} \tag{5.52}
\]

where
\[
\omega(\varphi, t) = \sum_{n = -\infty}^{\infty} p(\varphi + 2\pi t, t) \tag{5.53}
\]

(Ch. 15, 32)

Applying the boundary conditions
\[
\omega(\pm \pi) = \omega(-\pi) \tag{5.54}
\]

and
\[
\int_{-\pi}^{\pi} \omega(\varphi) \, d\varphi = 1 \tag{5.54}
\]

it is easily shown that for \( \Omega = 0 \),
\[
\omega(\varphi) = \frac{2\pi}{\beta N_2} \left(\varphi \cos \varphi\right) \tag{5.55}
\]

where
\[
\alpha = \frac{4E}{\beta N_2} \tag{5.56}
\]
Thus, it is seen that the $p$-process is no longer Gaussian, but has a null mean. The variance of $\theta$ is given by

$$\sigma^2 = \frac{3}{\pi} \int \varphi(u) \, du$$

when $n$ is large, Eq. (5.55) reduces to

$$\sigma^2 = \frac{1}{2n} \exp \left( -\frac{\pi^2}{2(1/n)} \right)$$

The variance of the phase process in this case becomes

$$\sigma^2 = \frac{1}{n} \frac{\pi}{2n} \exp \left( -\frac{\pi^2}{2(1/n)} \right)$$

In this connection it is worthwhile to note that the probability density function of the difference of phase between the input and output of a rectangular bandpass limiter of bandwidth $W$ is given by

$$\sigma(u) = \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{u}{W} \right)^2 \right]$$

which, for large $n$, approximates to

$$\sigma(u) = \frac{1}{\sqrt{2\pi}} \cos \frac{\pi}{2} \exp \left( -\frac{\pi^2}{2} \right)$$

Here

$$\rho = \frac{\pi^2}{2nW^2}$$

Therefore, the variance of the phase error at the output of the BPL is
This comparing (5.63) with (5.59) it is seen that performance of an ISO approximates, at large CNR, to that of a BFL of the bandwidth of a rectangular BFL, and an ISO are sketched in Fig. 5.1. This clearly indicates that both the distributions become identical for large values of α.

![Figure 5.1](image-url)  

The probability density distribution of the phases of an injection synchronized oscillator and a bandwidth limited output.

But when the detuning is finite, i.e., $\Omega \neq 0$ the steady state pdf of $\phi$ is given by

$$a(\phi) = C_\phi \exp \left( \frac{X_\phi}{\tilde{\alpha}} \right)$$

$$\times \sum_{n} \exp \left( -X \frac{\Omega}{\tilde{\alpha}} - n \cos \phi \right)$$

where

$$\tilde{\alpha} = \frac{2\pi \rho \Omega}{4N_0}, \quad X = \frac{\Omega}{\tilde{\alpha}}$$

(5.64)
and
\[
\frac{1}{C_0} = \int_0^{\pi/2} \int_0^{\pi/2} \exp \left[ X (\phi - \psi) + \sin \phi \cos \psi \right] \sin \phi \, d\phi d\psi
\]  
(5.65)

Putting \( \phi - \psi = \nu \) one gets
\[
\frac{1}{C_0} = \int_0^{\pi/2} \int_0^{\pi/2} \exp \left[ -X \nu + 2 \sin \frac{\nu}{2} \cos \left( \frac{\nu + 2 \phi}{2} \right) \right] \sin \phi \, d\phi d\psi
\]

That is,
\[
\frac{1}{C_0} = \int_0^{\pi/2} \exp (X \nu) \int_0^{\pi/2} \exp \left[ 2 \sin \frac{\nu}{2} \cos \left( \frac{\nu + 2 \phi}{2} \right) \right] \sin \phi \, d\phi d\psi
\]
\[
- \int_0^{\pi/2} \exp (-X \nu) \int_0^{\pi/2} \exp \left[ 2 \sin \frac{\nu}{2} \cos \left( \frac{\nu - 2 \phi}{2} \right) \right] \sin \phi \, d\phi d\psi
\]

i.e.,
\[
\frac{1}{C_0} = 2\pi \int_0^{\pi/2} \exp (-X \nu) I_0 \left( 2 \sin \frac{\nu}{2} \right) \, d\nu
\]  
(5.66)

or
\[
\frac{1}{C_0} = 4\pi \int_0^{\pi/2} \exp (-X \nu) I_0 \left( 2 \sin \nu \right) \, d\nu
\]  
(5.66a)

From this expression one can easily compute the probability that the system loses lock on the two sides of the free-running frequency of the oscillator. These are respectively given by,
\[
P_\nu = P_{\text{sub}} \left( \pi/2 - \phi_0 \leq \psi \leq \pi \right) = \int_{\pi/2 - \phi_0}^{\pi} u(\psi) \, d\psi
\]  
(5.67)

and
\[
P_\phi = P_{\text{sub}} \left( -\pi < \psi < -\pi/2 - \phi_0 \right) = \int_{-\pi - \phi_0}^{-\pi/2 - \phi_0} u(\psi) \, d\psi
\]  
(5.68)

where \( \phi_0 \) is the steady state phase difference corresponding to the
open-loop frequency error in the noise-free situation. Variations of $P_c$ and $P_t$ with $x$ are shown in Fig. 5.2 for a fixed value of the detuning of the carrier from the centre frequency of the oscillator for the following two cases: (i) centre frequency lying below the carrier frequency, and (ii) centre frequency lying above the carrier frequency. These plots clearly point out that, for a fixed frequency error, the probability of losing lock is greater when the carrier lies below the centre frequency than when it is located on the upper side of the centre frequency. This is because of the asymmetric looking
characteristics on the two sides of the cut-off frequency.

Now to calculate the pdf of the amplitude distribution, we recall that the amplitude fluctuation is small compared to that of the phase fluctuation, because of the limited type non-linearity of the oscillator. Thus we assume that at any instant

$$A = A_s (1 + \phi), \text{ where } \phi \ll 1$$

(5.69)

Thus it is to say, the perturbation is caused due to noise only. Moreover, it is relatively a slowly varying process compared to $A_s$. Thus the instrumental equation for the amplitude is given by (5.2.34):

$$\frac{dA}{dt} + \beta \langle A \rangle F(A, \phi) = \beta \langle A \rangle F(A, \phi) \frac{d\phi}{dt} = \phi F(A, \phi)$$

(5.70)

where

$$F(A, \phi) = -\frac{\beta}{2\sigma^2} (C_1) A^3 - \frac{1}{4} C_4 A^4$$

(5.71)

i.e.,

$$F(A, \phi) = \left[ (A_s / A)^3 - 1 \right] (C_1)$$

(5.71a)

$$A_s^4 = \frac{\sigma^2}{\beta C_1}$$

(5.72)

Applying the methods as indicated earlier, one computes

$$\mathcal{A}_s(t) = \lim_{\sigma \to +\infty} \mathbb{E}[\Delta z(t)/\Delta t]$$

$$= -\beta \langle A \rangle F(A, \phi, \phi) A_s$$

(5.73)

and

$$\mathcal{A}_s(t) = \lim_{\sigma \to +\infty} \frac{\partial <A^2>}{\partial t} \int_0^\infty \int_0^\infty N_2(N_1, N_2) \, dN_1 \, dN_2$$

$$= \left( \frac{\beta \langle A \rangle}{A_s} \right)^2 \frac{N_s}{2}$$

(5.74)

Thus the pdf of $\phi$ is given by

$$\frac{\sigma(x)}{\sigma} = -\frac{1}{2} \frac{\beta \langle A \rangle}{\sigma} \mathbb{P}(\phi, \phi) F(A, \phi) \frac{d\phi}{dt} = \phi F(A, \phi)$$

(5.75)

In the steady state one gets
Thus it is easily seen that
\[
p(a) = \frac{\exp\left(-\frac{a^2}{2\sigma^2}\right)}{\sqrt{2\pi}\sigma^2}
\]
(5.77)
where
\[
\sigma^2 = \frac{\langle A^2 \rangle}{4F_a A_0^2}
\]
(5.78)
Thus the perturbed amplitude is seen to have a Gaussian distribution with a variance \(\sigma^2\) and null mean.

5.5.2 Power Spectral Density at the Output
Let us suppose that the oscillator is in tune with the synchronizing signal. Therefore, the output is given by
\[
y(t) = A_x(1 + a\cos(\omega t - \phi))
\]
(5.79)
Hence the output auto-correlation function is given by
\[
R_y(\tau) = E[y(t)\cdot y(t + \tau)]
= E[A_x^2(1 + a\cos(\omega t - \phi))\cos(\omega(t + \tau) - \phi)]
= \cos(a\omega(t + \tau) - \phi)
\]
(5.80)
where
\[
R_y(\tau) = E[y(t + \tau)\cdot y(t)]
\]
(5.81)
Now let us suppose that carrier-to-noise ratio is not very small so that one can approximate \(p(\phi)\) as
\[
p(\phi) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{\phi^2}{2\sigma^2}\right)
\]
(5.82)
Thus \(p(\phi)\) appears to be Gaussian with zero mean. Putting \(\phi(t + \tau) = \phi_1\) and \(\phi(t) = \phi_1\) we noting that \(p(\phi)\) is Gaussian one has
\[
E[\cos(\phi(t + \tau) - \phi(t))] = E[\cos(\phi_1 - \phi_1)] = 0
\]
\[ R_d = E \exp \left( j \left( 2\pi f_d T - \phi_d \right) \right) \]  
where \( R_d \) denotes "the real part of \( R_d \)." Note that

\[ E \exp \left( j \left( 2\pi f_d T - \phi_d \right) \right) \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( j \left( 2\pi f_d T - \phi_d \right) \right) \frac{1}{2\pi} \text{d}\phi \text{d}T\]

\[ = \exp \left\{ - \frac{1}{2} \left( u_{11} Z_1^2 + u_{12} Z_1 Z_2 + \frac{1}{2} u_{22} Z_2^2 \right) \right\} \]  

(5.83)

where

\[ u_{11} = u_{22} = \frac{\theta}{2} \quad \text{and} \quad u_{12} = R_d(\theta) \]

(5.85)

Thus

\[ R_d = \frac{A_d^2}{2} \exp \left\{ - R_d(0) \right\} \cos \theta \sin \phi \cos \varphi \]  

(5.88)

Now for most of the cases \( R_d(\theta) \ll 1 \) and \( R_d(\phi) \ll 1 \) so that (5.88) reduces to

\[ R_d = \frac{A_d^2}{2} \exp \left\{ - R_d(0) \right\} \cos \theta \sin \phi \cos \varphi \]  

(5.89)

\[ R_d(\theta) \text{ and } R_d(\phi) \text{ can be found by referring to (5.70) and (5.35a).} \]

Linearise the phase and amplitude equations and write

\[ \frac{d\psi}{dt} = M \psi \]

(5.90)

\[ \frac{dA}{dt} = \frac{1}{2} \left( A + B \right) R_A \]  

(5.91)

Thus the spectral densities of the phase difference and amplitude perturbation are respectively given by

\[ S(\psi) = \frac{\beta^2 N_r^2}{\omega^2 + (\beta \psi)^2} \]  

(5.92)

\[ S(A) = \frac{1}{2} \omega^2 + (A \psi)^2 \]  

(5.93)
Hence
\[ R_s(t) = \frac{\alpha N_r^2}{4\beta^2} \exp\left(-\beta B \mid \tau\right) \] (5.94)
and
\[ R_s(t) \sim \frac{\langle A \rangle N_r}{4A^2} \exp\left(-\frac{\langle A \rangle}{F(A)} | \tau | \right) \] (5.55)

Therefore, the spectral density of the oscillator output is
\[
\Phi(\omega) = \int R_s(\tau) \exp\left(-j\omega \tau\right) \, d\tau
= \frac{\alpha^2}{2} \exp\left(- R_s(0) \right) \left[ \int \exp\left(-j\omega T \right) \cos \omega_0 \, d\tau + \int R_s(\tau) \exp\left(-j\omega \tau \right) \cos \omega_1 \, d\tau \right] \] (5.56)

Note that
\[
\int \exp\left(-j\omega \tau \right) \cos \omega_0 \, d\tau = \left[ \int \exp\left(j(\omega_0 - \omega) \tau \right) \, d\tau \right] \\
+ \frac{1}{2} \left[ \int \exp\left(-j(\omega_0 + \omega) \tau \right) \, d\tau \right]
= \pi \delta(\omega_0 - \omega) \] (5.57)

\[
\int R_s(\tau) \exp\left(-j\omega \tau \right) \cos \omega_1 \, d\tau
= R_s(0) \left[ \int \exp\left(-b \mid \tau\right) \exp\left(-j\omega \tau \right) \, d\tau \right] + \exp\left(-j\omega \tau\right) \] (5.58)

where
\[ b = \delta \langle A \rangle F(A) \]
\[
\begin{align*}
  & \int \mathcal{R}_0(\tau) \exp (-j\omega \tau) \cos \omega_0 \tau \, d\tau \\
  = & \frac{b\mathcal{R}_0(0)}{b^2 + (\omega - \omega_0)^2} + \frac{\mathcal{R}_0(0)}{(b^2 + (\omega + \omega_0)^2)} \\
  & + \int \mathcal{R}_0(0) \frac{bE}{(b^2 + (\omega - \omega_0)^2)} \exp (-j\omega \tau) \cos \omega_0 \tau \, d\tau \\
  & \quad \quad \quad + \mathcal{R}_0(0) \frac{bE}{(b^2 + (\omega + \omega_0)^2)} \\
  & = \mathcal{R}_0(0) \left( \frac{bE}{b^2 + (\omega - \omega_0)^2} + \frac{bE}{b^2 + (\omega + \omega_0)^2} \right) \\
  & \quad \quad + \left[ bR(\omega - \omega_0) + \frac{bR(\omega + \omega_0)}{b^2 + (\omega + \omega_0)^2} + \frac{bR(\omega - \omega_0)}{b^2 + (\omega - \omega_0)^2} \right] \\
  & = \mathcal{R}_0(0) \left( \frac{bE}{b^2 + (\omega - \omega_0)^2} + \frac{bE}{b^2 + (\omega + \omega_0)^2} \right) \\
  & \quad \quad + \left[ bR(\omega - \omega_0) + \frac{bR(\omega + \omega_0)}{b^2 + (\omega + \omega_0)^2} + \frac{bR(\omega - \omega_0)}{b^2 + (\omega - \omega_0)^2} \right]
\end{align*}
\]
\begin{align*}
\text{The output CNR is}
\text{(CNR)}_o &= \frac{A_1^2/2 \exp \left(-\frac{b_N}{4E} \right)}{A_s^2/2 \exp \left(-\frac{b_N}{4E} \right)} \left( \frac{B_v}{4E} \right) \left( \frac{A_+ N + A_0^2}{4A_s^2 F(A_s)} \right) \tag{5.102}
\end{align*}

The bandwidth of the predetection filter is \(W\), and thus one finds that the input CNR is given by
\begin{align*}
\text{(CNR)}_i &= \frac{S^2}{N_0 W} - \frac{S^2}{2 R_s W}
\end{align*}

Thus
\begin{align*}
\text{(CNR)}_i &= \frac{S^2}{N_0 W} - \frac{S^2}{2 R_s W} \\
&\quad \frac{B_v}{4E} \left( \frac{A_+ N + A_0^2}{4A_s^2 F(A_s)} \right) \tag{5.103}
\end{align*}

Thus by lowering the value of \(bE/4\) in comparison to \(W\), the output CNR can be much improved. Further a judicious choice of the nonlinearity will also help improving the output CNR.

5.6 Remarks

This Chapter presents the synchronization characteristics of an injection locked oscillator for a signal corrupted with an interfering tone or additive white Gaussian noise. For the case of low interference, the locking characteristics of the oscillator can be evaluated.
by considering a shift in the frequency of oscillation due to the interference. Moreover, because of the asymmetry in the tuning characteristics of the oscillator, probability of locking being on the two sides of the entire frequency have been found to be different. By a judicious choice of the hold-in range and the asymmetry in the spectral purity property of an oscillator can be enhanced. This is quite different from that of a phase-locked loop. It is interesting to study the synchronization property of the oscillator when the incoming signal is contaminated with both the interfering tones and the additive white Gaussian noise.

REFERENCES

6.1 Introduction

In this chapter we will discuss the amplifying and filtering properties of an oscillator in response to an FM signal contaminated with additive random noise. Discussion will be restricted to the case of low level synchronization. This is justified because the strength of the received signal is usually small and it needs to be small in comparison with the free running amplitude of the oscillator for reasons of better performance as discussed in the earlier chapters. These properties of an oscillator have been studied by many authors [1-6].

Incidentally, it is to be emphasized that the entire performance of an injection locked oscillator, either as a narrowband filter or as an amplifier, depends on its capability of tracking the signal. Thus, the locking range is an important parameter. In view of this, we will first evaluate locking bandwidth, and afterwards methods for distortion evaluation will be introduced.

6.2 Locking Bandwidth for a Noise-free FM Signal

For the sake of convenience we rewrite (4.22) and (4.23) as

$$\frac{d\delta}{dt} = \frac{\alpha_0}{2\eta} (1 - e^\Omega) a + \frac{\alpha_0}{2\eta} F \cos \varphi \quad \quad (6.1)$$

$$\frac{d\varphi}{dt} = \Omega - \frac{\alpha_0}{2\eta} F \sin \varphi - \frac{d\delta}{dt} \quad \quad (6.2)$$

It is seen that evaluation of the oscillator performance requires simultaneous solution of (6.1) and (6.2). Incidentally it may be mentioned that in section 4.3, we have given a method of evaluating
the locking range when the strength of the signal is small compared to that of the oscillator. However, if the strength of the incoming signal is neither small, nor very large, then the amplitude fluctuation in the phase equation can be taken care of by putting (cf. section 3.3)

\[ a \approx 1 + \frac{F}{2} \cos \varphi \]  

(6.3)

in (6.2) and rewrite (6.2) as

\[ \frac{\partial}{\partial t} \hat{Q} = \frac{\omega_0}{2F} F \sin \varphi \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \varphi} \]  

(6.4)

Putting

\[ F = \Delta \]  

and

\[ K = \frac{\omega_0}{2F} F \]  

(6.5)

one gets

\[ \frac{\partial}{\partial t} \hat{Q} - K \frac{\partial}{\partial \varphi} \frac{\sin \varphi}{1 + \frac{1}{\Delta} \cos \varphi} = \frac{\partial}{\partial \varphi} \]  

(6.6)

Let us now suppose that the oscillator is under the influence of a single tone FM signal, i.e.,

\[ \Phi = \frac{\Delta}{\omega_0} \sin \omega_0 t \]  

(6.7)

Further assume that the oscillator is operating under locked condition with an average phase error \( \varphi_0 \), which is a measure of the initial frequency error in relation to the maximum locking range \( \Delta \).

Thus we assume the solution of (6.6) as

\[ \varphi = \varphi_0 + M \sin (\omega_0 t + \delta) \]  

(6.8)

where \( M \) is the input-output index error.

Locking of an oscillator to an FM signal means that the average value of the instantaneous frequency error is zero, i.e.,

\[ \int_{\text{cycles}} \frac{\partial \varphi}{\partial t} = 0 \]  

(6.9)

Putting

\[ F = \Delta \]  

(6.10)

\[ r = \frac{1}{2} - \sqrt{\frac{1}{\Delta^2}} - 1 \]  

(6.11)
and expanding

$$\sin \phi \over 1 + x \cos \phi$$

in a Fourier series, (cf. section 3.4) we rewrite (6.5) as

$$\frac{d\phi}{dt} = \Omega - \frac{2a_0}{q} \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \cos \left( n \phi \right)$$

(6.12)

i.e.,

$$\frac{d\phi}{dt} = \Omega - \frac{2a_0}{q} \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} \cos \left( n \phi \right)$$

(6.13)

Using (6.7) through (6.13) one finds that the locking range is given by

$$\Omega = \frac{2a_0}{q} \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} n \sin \left( n \phi \right)$$

(6.14)

Referring to Chapter 3 (cf. 3.84) one finds that in the absence of modulation, the maximum value of $\phi_0$ is seen to be

$$\phi_0 = n/2 + M \sin \left( n/2 \right)$$

(6.15)

Again applying the method of harmonic balance (cf. Chapter 4) it is easily found that

$$M = \phi_0 + \frac{2a_0}{q} \sum_{n=1}^{\infty} \left( -1 \right)^{n+1} n \phi_0$$

(6.16)

Thus using (6.14), (6.15) and (6.16) one can find the locking range of an oscillator in response to an FM signal.

6.3 Synchroonization for a Noisy FM Signal

In this case the system equations are given by (5.34) and (5.35).

Since in most of the cases, the strength of the synchronizing signal is small compared to that of the oscillator, we will not consider an over-driven oscillator, instead an under-driven oscillator will be considered. Thus we will ignore the amplitude equation i.e., we neglect amplitude fluctuation i.e., $M(t) = A(t)$ and rewrite (5.35) as
\[ \frac{d\varphi}{dt} = \Omega - \beta E \sin \varphi + \beta h_0(t) + \frac{d\theta}{dt} \]  
(6.17)

where
\[ \beta = \frac{c_0}{2D_0} \]  
(6.18)

Locking of an ISO with respect to an FM signal contaminated with an additive random noise will be said to occur, when the average instantaneous frequency error is zero, i.e., \( \langle \frac{d\varphi}{dt} \rangle = 0 \). Before attempting to evaluate the locking range as defined earlier, let us start with the assumption that the oscillator is in the locked state under the influence of the signal. Assuming an initial deviation, it is not hard to conjecture that the instantaneous phase error will consist of three terms, viz., (i) a dc component due to the initial detuning, (ii) a component at the modulating frequency, and (iii) random fluctuations due to the noise. As a result, the solution (cf. 6.17) may be assumed to be of the form
\[ \varphi = \varphi_0 + M \sin(\omega_M t + \beta) + \varphi(t) \]  
(6.19)

where \( \varphi_0(t) \), having zero mean, is the random variable due to the incoming noise. Taking the time average of (6.17) one can write
\[ \langle \frac{d\varphi}{dt} \rangle = \Omega - \beta E \langle \sin \varphi \rangle + \beta \langle N_0(t) \rangle + \langle \frac{d\theta}{dt} \rangle \]  
(6.20)

Assuming a sinusoidal angle modulation and remembering that \( \langle \frac{d\theta}{dt} \rangle = 0 \) in the locked state, the locking range of the ISO in such a situation is given by
\[ \Omega = \beta E \langle \sin \varphi \rangle \]  
(6.21)

To find \( \langle \sin \varphi \rangle \) let us remember that \( \varphi \) consists of a dc part \( \varphi_0 \), an alternating part \( M \sin(\omega_M t + \beta) \) and a random Gaussian variable \( \varphi_0(t) \) with variance \( \sigma_0^2 \) as given by (6.19). Note that actually \( \varphi_0 \) is not truly Gaussian but if the input carrier-to-noise ratio is not very small the probability density function of \( \varphi_0 \) approximates closely to that of a Gaussian variables (cf. Chapter 5). Thus one can write
\[ p(\varphi) = \frac{1}{\sqrt{2\pi} \sigma_0} \exp \left( -\frac{\varphi^2}{2\sigma_0^2} \right) \]  
(6.22)

Thus
\[ \sin \varphi = \sin [\varphi_0 + M \sin(\omega_M t + \beta) + \varphi_0] \]
\[\cos \left( \sin (\omega t + \theta) \right) = \frac{1}{2} \left[ \cos (\cos (\phi) \sin (\omega t + \theta)) + \cos (\phi) \sin (\omega t + \theta) \right] \]  
and  
\[\sin \left( \sin (\omega t + \theta) \right) = \frac{1}{2} \left[ \sin (\cos (\phi) \sin (\omega t + \theta)) + \sin (\phi) \sin (\omega t + \theta) \right] \]

Therefore, (6.27) reduces to
\[\sin \phi = J_0(\omega t) \sin \phi \cos \phi + \cos \phi \sin \phi \]

Now one has
\[\sin \left( \sin (\omega t + \theta) \right) = \frac{1}{2} \left[ \cos (\cos (\phi) \sin (\omega t + \theta)) + \cos (\phi) \sin (\omega t + \theta) \right] \]  
and  
\[\cos \left( \sin (\omega t + \theta) \right) = \frac{1}{2} \left[ \sin (\cos (\phi) \sin (\omega t + \theta)) + \sin (\phi) \sin (\omega t + \theta) \right] \]

Thus (6.28) reduces to
\[\sin \phi = J_0(\omega t) \sin \phi \exp \left( - \frac{\phi^2}{2} \right) \]

Hence (6.29) can be written as
\[
\Omega = \beta J_0(M) \sin \varphi_0 \exp \left( - \frac{a^2}{2} \right) \quad (6.32)
\]

The above relation gives the synchronization band of the oscillator provided \( M, \varphi_0 \), and \( a^2 \) are known. Obviously, the locking band corresponds to the maximum allowable value of \( \varphi_0 \). This is given as

\[
(\varphi_{\text{max}} = \pm \left( \frac{\Omega}{2} - M - \beta \right) \quad (6.33)
\]

Plus and minus signs denote respectively the upper or lower side locking range of the oscillator. To find the value of \( M \) and \( a^2 \) we rewrite the locking equation (6.17) by substituting the assumed solution of \( \varphi \) from (6.19) as

\[
\frac{d^2 \varphi}{dt^2} + \frac{M}{M_{\text{in}}} \cos (\omega a + \varphi) + \frac{dM}{dt} = \Omega - \beta E \sin (\varphi_0) \\
+ \frac{M_{\text{in}}}{M} \sin (\omega a + \varphi_0) + \frac{dM_{\text{in}}}{dt} + \frac{d\varphi_0}{dt} \quad (6.34)
\]

Let,

\[
\frac{d^2 \varphi}{dt^2} + \frac{M}{M_{\text{in}}} \cos (\omega a + \varphi) + \frac{dM}{dt} = \Omega - \beta E \sin (\varphi_0) \cos \varphi_a \\
+ \cos \varphi_0 \sin \varphi_0 \frac{d^2 J_0}{dx^2} (M) \cos \varphi \cos \varphi_a + \cos \varphi_0 \cos \varphi_0 \cos \varphi_a \\
- \sin \varphi_0 \sin \varphi_0 \frac{d^2 J_0}{dx^2} (M) \sin \varphi \cos (\omega a + \varphi) \\
+ \frac{dM_{\text{in}}}{dt} + \frac{d\varphi_0}{dt} \quad (6.35)
\]

where, \( J_0 (x) \) is the Bessel's function of the first kind with order \( n \) and argument \( x \). The complexity of the nonlinear differential equation (6.35) suggests that some approximate nonlinear analytical technique is to be applied to (6.25) for making it amenable to solution. An extremely powerful technique is to apply the statistical linearization \([7, 8]\) principle to the nonlinear components (viz., \( \sin \varphi_0 \) and \( \cos \varphi_a \)). According to this principle any nonlinear function \( f(x) \) without exception can be replaced by some linear function, that is,

\[
f(x) = a_0 + a_1 x \quad (6.36)
\]

where\

\( a_0 \) and \( a_1 \) are given by
\[ a_t = \int f(x) p(x) \, dx \]  
\[ a_0 = \frac{\int x f(x) p(x) \, dx}{\int x f(x) \, dx} \]  
(6.37)  
\[ \text{where} \quad p(x) \text{ is the probability distribution function of the variable 'x'. Therefore, } \sin \varphi_a \text{ and } \cos \varphi_a \text{ can be approximated by the following relation:} \]
\[ \sin \varphi_a = \exp \left( -\frac{\Delta}{2} \right) \]  
\[ \cos \varphi_a = \exp \left( -\frac{\Delta}{2} \right) \]  
(6.39)  
\[ p(\varphi_a) \text{ is assumed to be Gaussian with a variance } \Delta. \]
Comparing (6.35), (6.39) and (6.40) and applying the method of harmonic balance with \( \theta = \Delta \cos \varphi_a \) one has
\[ M_{\text{opt}} = \Delta \cos \theta \]  
\[ 0 = \Delta \sin \theta - \beta E \frac{a_t}{2} \cos \phi_b \exp \left( -\frac{\Delta}{2} \right) \]  
(6.41)  
\[ \text{and} \]
\[ M^2 = a_0^2 + \frac{\beta^2 E^2}{4} \Sigma^2 \exp \left( -\frac{\Delta}{2} \right) \]  
(6.42)  
\[ \text{Therefore, from (6.41) and (6.42) one gets} \]
\[ M^2 = N \theta - \frac{\beta E}{2} a_t \]  
\[ \theta = \frac{\beta E a_t}{2} \]  
(6.43)  
\[ \text{and} \]
\[ M^2 = N \theta - \frac{\beta E}{2} a_t \]  
\[ \theta = \frac{\beta E a_t}{2} \]  
(6.44)  
\[ = \frac{1}{2 \pi} \int_0^{\pi} \frac{1}{4} \cos \varphi_a \]  
(6.45)
where \( p = \frac{I_s}{2} \) represents the carrier-to-noise power ratio.

Hence combining (6.32), (6.33) and (5.54), the expression for the locking ratio \( R = f(\theta) \) is given by

\[
R = J_0^2 \left( \delta \right) \exp \left( -\frac{\delta^2}{2} \right) \left( 1 + \frac{1}{\delta^2} \right)
\]

(6.46)

The above relation gives the variation of the locking ratio with the frequency deviation of the signal and the carrier-to-noise. The variation of \( R \) with the CNR is shown in Fig. 6.1.

---

6.4. Filtering Property

Narrowband tunable filtering property of an injection-synchronized oscillator for a pure angle-modulated signal has been studied in chapter 4. In this section the capability of an injection-synchronized oscillator in purifying an angle-modulated signal contaminated with additive random disturbances will be judged. Particularly, the property of an ISO in generating a reference carrier from a corrupted signal will be reviewed critically. In such a situation it is impos-
tively clear that the modulating frequency tends to be much larger than the locking bandwidth of the ISO (cf. 4.41).

For the sake of simplicity, let us consider that the carrier frequency of the ISO is equal to the carrier frequency of the modulated signal. Let us further assume that the incoming carrier-to-noise ratio is such that locking is achieved on the average. In such a situation the phase error may be assumed to be given by

$$\eta = \eta_n + \Phi$$

(6.47)

where \(\eta_n\) and \(\eta\) are the phase errors due to the modulation and the noise processes respectively. Applying the quantization technique and utilizing (6.17), the instantaneous fluctuations due to the modulation and the noise processes are easily shown to be

$$\frac{d\eta}{dx} = - \frac{1}{F} \exp \left( - \frac{x^2}{2} \right) \sin \eta_n + \frac{D}{F} \cos x$$

(6.48)

$$\frac{d\Phi}{dt} = - \pi F \exp \left( - \frac{x^2}{2} \right) \cos \eta_n + \beta N_1(t)$$

(6.49)

where

$$x = x_{eq}$$

$$\beta = m \sin \omega_{eq}$$

$$\frac{D}{\beta x} F = \frac{\omega}{\pi} \beta E - \frac{\omega \beta E}{2 Q L_n}$$

and \(D^2\) represents the variance of the phase error due to the noise process. Now to have a solution of the equations (6.48) and (6.49) we first replace \(\eta_n\) by \(\eta\) and rewrite (6.48) as first order approximation,

$$\frac{d\eta}{dx} + \frac{D}{F} \eta = \frac{D}{F} \cos x$$

(6.50)

where

$$B = \frac{1}{F} \exp \left( - \frac{x^2}{2} \right)$$

Hence the solution of (6.50) is given by,

$$\eta_n = \frac{D}{\pi B} \left[ \beta \cos x + \sin x \right]$$

(6.51)

Now to have the second order solution, take,

$$\eta_n = \eta_n + \eta$$

(6.52)
where \( \eta_m \) is a small quantity such the \( \sin \eta_m = \eta_m \) and \( \cos \eta_m = 1 \).

Hence eqn. 6.50 gets from

\[
\frac{d^2 \eta_m}{dx^2} = \frac{D}{1 + \frac{D}{F}} \left\{ -B \cos x + \sin x \right\} + \frac{d \eta_m}{dx}
\]

(6.53)

and

\[
\sin \eta_m = 2J_0 \left( \frac{D}{F} \cdot B \right) J_0 \left( \frac{D}{F} \cdot 1 \right) \cos x
\]

\[
+ 2J_1 \left( \frac{D}{F} \cdot B \right) J_1 \left( \frac{D}{F} \cdot 1 \right) \sin x
\]

\[
+ J_2 \left( \frac{D}{F} \cdot B \right) J_2 \left( \frac{D}{F} \cdot 1 \right) \eta_m
\]

(6.54)

Thus the equation for the second order approximation looks like,

\[
\frac{d^2 \eta_m}{dx^2} + P \frac{d \eta_m}{dx} = Y \cos x + Z \sin x
\]

(6.55)

where,

\[
T = B \cdot J_0 \left( \frac{D}{F} \cdot 1 \right) J_0 \left( \frac{D}{F} \cdot B \right)
\]

\[
Y = D - \frac{D}{F} - 2B J_1 \left( \frac{D}{F} \cdot B \right) J_0 \left( \frac{D}{F} \cdot 1 \right)
\]

\[
Z = \frac{D}{F} \cdot B - 2B J_0 \left( \frac{D}{F} \cdot B \right) J_1 \left( \frac{D}{F} \cdot 1 \right)
\]

Therefore,

\[
\eta_m = \left( \frac{P + \frac{Y}{1 + \frac{Z}{P^2 + 1}}} \right) \cos x
\]

\[
+ \left( \frac{Y + \frac{Z}{P^2 + 1}} \right) \sin x
\]

(6.56)

Hence,

\[\eta_m = m_n \cos x + m_n \sin x \]

(6.57)

where

\[
m_n = \frac{mB + \frac{P}{1 + \frac{Z}{P^2 + 1}}}{1 + \frac{1}{P^2 + \frac{Z}{1 + \frac{P}{1 + \frac{Z}{P^2 + 1}}}}}
\]

and

\[
m_n = \frac{m}{1 + \frac{1}{P^2 + \frac{Z}{1 + \frac{P}{1 + \frac{Z}{P^2 + 1}}}}}
\]
The instantaneous phase $\Psi(t)$ of the output of the ISO will then be of the form

$$\Psi(t) = (m - m_n) \sin \omega t - m_n \cos \omega t - \Psi_0(t)$$  \hspace{1cm} (6.58)

It is, therefore, easy to find the carrier power at the output of the ISO as given by

$$\text{Carrier power} = \frac{1}{4} A^2 \left| (m - m_n) J_0^2 (m_n) \exp (- \frac{\Psi_0}{2}) \right|$$  \hspace{1cm} (6.59)

From the conservation of power, the sideband power at the output of the ISO is given by

$$\text{Sideband power} = \frac{1}{2} A^2 \left| 1 - J_2 (m - m_n) J_2^*(m_n) \exp (- \frac{\Psi_0}{2}) \right|$$  \hspace{1cm} (6.60)

The figure of merit of the ISO in generating a carrier erase from a noisy angle modulated signal is defined as the ratio of the carrier- to-sideband power at the output of oscillation. The figure of merit ($\eta$) will have the following expression

$$\eta = \frac{J_2^2 (m - m_n) J_2^2 (m_n) \exp (- \frac{\Psi_0}{2})}{1 - J_2^2 (m - m_n) J_2^2 (m_n) \exp (- \frac{\Psi_0}{2})}$$  \hspace{1cm} (6.61)

Correspondingly the input carrier-to-noise power ratio ($\chi$) will maintain the relation

$$\chi = \frac{1}{2 \Delta f} (m_n) J_0^2 (m_n)$$  \hspace{1cm} (6.62)

For $\Delta f \gg 1$ (i.e., the modulating frequency is large compared to the locking range) (6.61) and (6.62) may be approximately rewritten in the form

$$\eta = \frac{\chi^2}{2 \Delta f} \exp \left( - \frac{\chi^2}{2 \Delta f} \right) \frac{J_2^2 (m_n) \exp \left( - \frac{\Psi_0}{2} \right)}{1 - J_2^2 (m_n) \exp \left( - \frac{\Psi_0}{2} \right)}$$  \hspace{1cm} (6.63)

and

$$\chi = \frac{1}{2 \Delta f} \exp \left( - \frac{\chi^2}{2 \Delta f} \right)$$  \hspace{1cm} (6.64)

respectively. The variation of the figure of merit ($\eta$) with the noise-to-carrier-power ratio ($\chi$) has been depicted in Fig. 6.2. The filtering property of the ISO gradually deteriorates as the modulating
frequency becomes comparable to or less than the locking band of the ISO. Thus, to realise all the advantages of generating a reference carrier from a noisy FM signal, the locking range of the ISO should be kept small compared to the modulating signal.

6.5 Amplifying Property

Referring to (6.43) through (6.45) it is not difficult to see that the major requirements for an injection locked amplifier are: (i) the locking range should be large compared to the highest frequency of the modulating signal, and (ii) the locking range should be adequate so that it does not lose lock. These mean that for a definite value of the input carrier-to-noise power ratio the power gain is limited. The nonlinear and delay distortion characteristics of the injection locked amplifier have been studied in Chapter 4 for a noise-free angle-modulated signal. In this section distortions due to the accompany-
ing noise will be considered. If the incoming signal-to-noise ratio is not small the net input to the ISO may be written as
\[ e(t) = [E + N_0(t)] \cos (\omega_0 t + \theta(t)) \cos \frac{1}{2} \theta(t) \]

(6.65)

where \( \theta(t) \) is the equivalent angle modulation due to noise and is approximately equal to \( N_0(t)/E \). Therefore, the governing phase equation of the ISO is
\[ \frac{d\varphi}{dt} = \frac{\omega_0 E + N_0(t)}{2Q A_0} \sin \varphi + \frac{1}{2} \frac{d\theta}{dt} \]

(6.66)

We have ignored the amplitude fluctuation of the oscillator.

In order to solve (6.66), the method of integration will be adopted. For a high input carrier to noise ratio, let us first assume that
\[ \frac{\omega_0 E + N_0(t)}{2Q A_0} \approx \frac{\omega_0 E}{2Q A_0} = K \]

In the next step, correction due to the term \( \frac{N_0(t)}{2Q A_0} \) will be introduced. With this in mind, let us replace \( \sin \varphi \) by \( \varphi \) and obtain the following equation
\[ \frac{d\varphi}{dt} + K \varphi = \omega_\text{in} \cos \omega_f t + \frac{1}{2} \frac{d\theta}{dt} \]

(6.67)

Now representing \( \frac{d\varphi}{dt} \), the random modulation term, in the series form as
\[ \frac{d\varphi}{dt} = \sum c_n \cos (\omega_f n + \varphi) \]

(6.68)

Thus using (6.67) and (6.68), it is shown that the first order solution is given by
\[ \varphi_1 = \left( \omega_\text{in} \cos \omega_f t + \frac{1}{2} \frac{d\theta}{dt} \right) K \]

(6.69)

which may be put as
\[ \varphi_1 = C_0(t) + C_1(t) \]

(6.70)

where
\[ C_0(t) = \frac{\omega_\text{in}}{K} \cos \omega_f t \]
and

\[ C_d(t) = \frac{1}{K} \frac{d\theta_5}{dt} \]

For the second order solution let us approximate \( \varphi \) by \( \varphi \approx \frac{\varphi_0}{3} \).

Thus noting (cf. 6.69) that

\[ m \cos \varphi \frac{d^2 \varphi}{dt^2} + \frac{d^2 \varphi}{dt^2} = K \varphi_1 \]

we find from (6.66)

\[ \frac{d^2 \varphi}{dt^2} + K \varphi_1 \approx K(C_5 + C_1) + \frac{K}{6} (C_4 + C_5) \]

the solution of which is given by

\[ \varphi_1 = (C_5 + C_1) + \frac{1}{6} (C_4 + C_5) \]

(6.72)

Therefore, taking the effect of the term \( \frac{N(t)}{k^2} \frac{C_0(t)}{A_0} \), the approximate solution of (6.66) can be written as

\[ \varphi = C_d(t) + C_d(t) + \frac{1}{6} (C_4(t) + C_5(t) + C_4(t)) \]

(6.73)

When the locking range is large compared to the modulating frequency, it is easily shown that the instantaneous phase modulation of the oscillator is given by

\[ \varphi(t) = m \sin \theta_5 \left( t - \frac{1}{K} \right) \]

(6.74)

The first term of (6.74) indicates that the angle of modulation of the input signal is delayed by \( \frac{1}{K} \) seconds. The second term is a linear distortion due to noise. Third term signifies nonlinear distortions and the fourth term introduces cross modulation distortions. The nonlinearity of the device has negligible effect on the distortion of the output waveform. Since the noise process has been assumed to have mean zero, the total distortion power at the output of the ISO is given by
\[ \frac{\Delta^2}{N_0} = \frac{\Delta^2}{N_0} + \frac{2}{\pi} F(1 + m) \left( \frac{\Delta^2}{N_0} + \frac{1}{4} (mF)^2 \right) \]  

(6.73)

\[ \frac{\Delta^2}{N_0} = \frac{2}{\pi} F(1 + m) \]  

where \( \Delta^2 \) denotes the distortion power due to noise. In deriving (6.73), the Carson's rule (i.e., \( B = 2\omega(1 + m) \)) has been assumed to hold good at the input of the oscillator; \( F \) is the total input band. The variation of total distortion power (6.3) with the carrier-to-noise power ratio \( \delta \) has been shown in Fig. 6.3. It also recommends that

![Fig. 6.3: Variation of the total distortion power of an injection-synchronized oscillator with input carrier-to-noise ratio.](image)

the ISO will be a good amplifier for an FM wave, when its locking band is much larger than the highest frequency of the modulating signal.

6.6 Remarks

The new materials that are presented in this chapter, are concerned with the consideration of the amplifying property of an ISO in a noisy environment; quantization techniques have been utilized
to evaluate the locking characteristics of the class-A oscillator.
Obviously, the validity of the results depends on the assumption that the phase error is a Gaussian variable, which is not strictly true.
However, this is quite a reasonable approximation that leads to useful results.

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PHASE-LOCKED LOOP FUNDAMENTALS

In this chapter we describe the operation of a phase-locked loop in simple physical terms and develop the basic equation governing the behaviour of the phase-locked loop. We also present a very brief discussion on the properties of the loop components, viz., the phase detector, voltage-controlled oscillator and loop filter. We further present a method of analysing a first-order phase-locked loop and conclude the discussion with a design example.

7.1 Mechanization of the Phase-Locked Loop

The simplest form of a phase-locked loop is shown in Fig. 7.1. It consists of a phase-sensitive detector, a low-pass filter and a voltage controlled oscillator. Although in practice a more elaborate arrangement consisting of heterodyne loops, bandpass filters and acquisition aids is used, yet the behaviour of the simplified arrangements of practical phase-locked loops can be explained in terms of the basic model as shown in Fig. 7.1. However, in later chapters we will analyse typical practical phase-locked loops.

Fig. 7.1. The block diagram of a phase-locked loop.
The phase sensitive detector is a multiplicative device, the purpose of which is to produce an output voltage as a function of the instantaneous phase difference between the reference input and the output of the voltage controlled oscillator. The output of the phase detector is an odd periodic function of the phase difference. The voltage controlled oscillator (VCO), as the name implies, is an oscillator, the frequency of which can be modulated in accordance with an input voltage. Referring to the circuit configuration of Fig. 7.1, one finds that the phase locked loop (PLL) is an electronic feedback control device. It operates in the following way. To begin with, let us assume that the frequency of the reference input is equal to the center frequency of the voltage controlled oscillator. Therefore, the output of the phase detector, that depends upon the phase or the instantaneous frequency difference between the two signals, will be zero. Now if the frequency of the voltage controlled oscillator tries to drift, this change in frequency will be first felt by the phase detector as a phase difference, and it will produce a voltage in correspondence to a measure of the phase difference. This voltage will correct the frequency of the voltage controlled oscillator in such a way as to reduce the error in frequency to nothing. Similarly, if the frequency of the reference input changes by a certain amount then the output of the phase detector will shift the frequency of the VCO by the same amount. At this point it is to be noted that there is a limit to the frequency difference between the input and the output of the VCO up to which the frequency of the local oscillator can be controlled.

Let us now consider a situation when the frequency of the incoming signal is far away from the center frequency of the VCO. Observe the variation of the beat note, as detected at the output of the phase detector, with the variation of the input frequency. In such a case, one finds a curve as shown in Fig. 7.3. From the figure one finds that as one increases the frequency of the reference input, the beat frequency decreases almost linearly up to \( f_s \) at which it suddenly drops to zero. Thereafter it remains constant till the input frequency goes up to \( f_s \) from where it suddenly jumps to a finite value; after that the beat frequency increases almost linearly with the input frequency. If one now wishes to retrace the path by decreasing the frequency of the incoming signal, it is found that the beat note does not drop to zero at \( f_s \) but it does drop to nothing on further reduc-
Fig. 7.2. Best frequency characteristics of a phase-locked loop.

Finding the frequency, say at \( f_b \). On still further reducing, one finds that the zero beat frequency suddenly assumes a finite value after crossing \( f_r \). Thus one finds that there is a zone of hysteresis on each side of the centre frequency and this is shown by the shaded zone on Fig. 7.2. The frequency difference \((f_b - f_c)\) or \((f_b - f_r)\) is called the single-sided pull-in range of the PLL. Whereas the frequency difference \((f_b - f_c)\) or \((f_b - f_r)\) is called the pull-out range (single-sided) of the PLL. Although it is observed from Fig. 7.2 that when the frequency of the reference input is equal to \( f_b \), the beat angular frequency suddenly drops to zero but the PLL actually takes a finite time, called the locking time, to achieve synchronization. This can be visualized by referring to Fig. 7.3. The plot depicts the variation of the best frequency voltage with time. It shows that the beat frequency continuously decreases up to the transitional value from which it drops to zero. Locking carefully at Fig. 7.3 one finds that there are three distinct regions which the oscillator has to cross over before falling in synchronization with the reference input. The first one is the zone where the beat angular frequency drops to zero from the transitional beat frequency and is said to be the frequency pull-in region. The second one is the zone where the instantaneous phase of the oscillator attains a steady state.
value. The sum total of the time taken by the loop to go through the first and second regions is said to be frequency pulling time of the PLL, and the time taken during the phase pulling region is called phase pulling time of the PLL. Note that depending upon the damping factor of the loop, the phase difference may attain the steady state value monotonically in which case the loop is said to be over-damped; or the loop may execute a few transients before it settles down to the steady state, in which case the loop is said to be underdamped. This is also shown in Fig. 7.3. Analytical methods to evaluate the pull-in range and locking time will be developed in Chapter 10.

7.2 Operation of the Loop Components

In this section we will consider briefly the operation of the components of the phase-locked loop, viz., phase detectors, voltage controlled oscillators and low-pass filters. After this we will learn the mathematics of the phase locked loop.

7.2.1 PHASE DETECTORS

The circuit diagram of a typical phase detector is shown in Fig. 7.4. It consists of a centre tapped transformer, the ends of which are connected to two identical diodes $D_1$ and $D_2$ as shown, and the outputs of the diodes are filtered with the help of the resistance-
capacitance circuit. To the centre tap of the transformer is connected the other voltage through a transformer. Let us assume that the voltages \( v_1 \) and \( v_2 \) are of the forms

\[
v_1 = V_1 \sin (\omega t + \phi_1) \quad (7.1)
\]

\[
v_2 = V_2 \cos (\omega t + \phi_2) \quad (7.2)
\]

\( v_2 \) can also be written as

\[
v_2 = V_2 \cos (\omega t + \phi_2 - \theta) \quad (7.3)
\]

where

\[
\theta = (\phi_2 - \phi_1) t + \phi_1 - \phi_2 \quad (7.4)
\]

denotes the phase difference between \( v_1 \) and \( v_2 \). The voltage applied to the peak rectifier are

\[
v_{R1} = v_1 + v_2 \quad (7.5)
\]

\[
v_{R2} = v_3 - v_4 \quad (7.6)
\]

The dc voltage observed at the cathodes of the diodes are

\[
V_{R1} = |v_{R1}| \cos \theta \quad (7.7)
\]

and

\[
V_{R2} = |v_{R2}| \cos \theta \quad (7.8)
\]

where \( \theta \) is the conduction angle of the diodes and this is nearly zero. Therefore, the difference of potential between the cathodes of the diodes is

\[
\Delta V = V_{R1} - V_{R2} = |v_{R1}| - |v_{R2}| \quad (7.9)
\]

Note that

\[
1 v_0^2 = V_1^2 + V_2^2 + 2V_1V_2 \sin \phi \quad (7.10)
\]
and
\[ |v_{d0}|^2 = V_1^2 + V_2^2 - 2V_1V_2 \sin \phi \quad (7.11) \]

The variation of the phase detector output with the phase difference \( \phi \) for various values of \( V_2/V_1 \) is shown in Fig. 7.5. Note that for large values of \( V_2/V_1 \), the output becomes independent of \( V_2 \). This is usually the situation in practice where the output of the local oscillator is much larger than that of the incoming signal. Thus assuming that \( V_2 \gg V_1 \), one finds
\[ v_{d0} \approx V_1 + V_2 \sin \phi \]
\[ v_{d0} \approx V_1 - V_2 \sin \phi \]

Hence the potential of the centre point of the resistors \( R \) with respect to the ground is
\[ V_2 = 2V_1 \sin \phi \quad (7.12) \]
Thus the relation (7.12) indicates that the output of the phase detector is proportional to the sine of the phase difference between the two signals. Moreover, it is seen to be independent of the amplitudes of \( V_2 \). It is further to be noted that if either of the input voltages is
zero then the output of the phase detector is also zero. This is true provided the secondary of the transformer is a centre-tapped one. In this sense the phase detector is perfectly balanced. If the transformer is not perfectly balanced, an offset voltage will appear at the output, i.e., a dc voltage will appear even when one of the inputs is zero. A small fixed offset voltage may be balanced out by varying the output potentiometer. However, when the offset voltage is large, it is not possible to reduce it to zero with either of the inputs connected to the phase detector. Note that in such a case, if the offset voltage is greater than the actual dc output of the phase detector, $V_o$, the loop will not lock. Note further that if the transformer is not perfectly balanced, the offset voltage depends on the input carrier-to-noise ratio. As such at low carrier-to-noise ratio, the offset voltage will be excessive and the phase locked loop will be thrown out of lock. This may be called the phase detector threshold and it should not be confused with the loop threshold, to be discussed in Chapter 9. In practice, it is very difficult to realize a perfectly balanced phase detector and utmost care is to be taken in designing the centre-tapped transformer. Capacitive coupling between different parts of the winding is to be avoided. In view of this, the best result is achieved by winding the coil on a ferite core if the frequency is not very high.

Another type of phase detector, which is commonly used, operates on the following principle. The input signal is passed through a switching device, which is periodically saturated by the reference signal. The output of the switching device is low-pass filtered to get a voltage that depends on the phase difference between the two signals. This type of phase detector may be used up to about 100 MHz. However, for microwave frequency regime a balanced mixer using waveguide hybrid may be used.

Analog multipliers can be used to act as a phase detector provided the sum frequency component at the output of the multiplier is low-pass filtered. This type of phase detector is possible only at low frequency.

Aside the sine-type phase detector, the characteristics of which is given by (7.12), there are other types of phase detectors, such as, triangular, sawtooth, etc. A triangular phase detector can be realized by multiplying the outputs of two half limiters which are fed
by the two inputs. This is shown in Fig. 7.6. The outputs of the hard limiters are given by

\[
(v_{l, m+1})_{LUT} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2m+1} \sin(2m+1) \left( \cos \theta + \Psi \right)
\]

and

\[
(v_{l, m+1})_{LUT} = \frac{4}{\pi} \sum_{n=0}^{\infty} \left(-1\right)^n \frac{1}{2m+1} \cos(2n+1) \left( \cos \theta + \Psi \right)
\]

Therefore, the output of the low-pass filter is

\[
v_o = \frac{8}{\pi} \sum_{n=0}^{\infty} \left(-1\right)^n \frac{1}{(2m+1)^2} \sin(2n+1) \theta
\]

which is a Fourier series expansion of a repetitive triangular wave with a period of 2\pi as shown in Fig. 7.6. A sawtooth type of phase

![Block diagram of a triangular phase detector.](image)

Fig. 7.6. The block diagram of a triangular phase detector.

detector can be realized with the circuit of Fig. 7.7. Mathematical model of Fig. 7.7 is shown in Fig. 7.8. Hard limiter outputs of Fig. 7.8 are given by

![Block diagram of a saw-tooth phase detector.](image)

Fig. 7.7. The block diagram of a saw-tooth phase detector.
\[ (v_{2})_{m} = \frac{-\phi}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{2n+1} \sin(2\pi f + 1)(e_{n} + y_{n}) \]  
(7.16)

and

\[ (v_{2})_{m} = \frac{\phi}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{2n+1} \sin(2\pi f + 1)(e_{n} - y_{n}) \]  
(7.17)

The low-pass output after the multiplier is

\[ (v_{2})_{m} = \frac{\phi}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{2n+1}(2m+1) \cos(2\pi f + 1)y_{n} \]  
(7.18)

The amount of dc bias to be added is

\[ \frac{\phi}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{2n+1}(2m+1)^{2} = 1 \]  
(7.19)

Therefore, the output is given by

\[ (v_{2})_{m} = \frac{\phi}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{2n+1}(2m+1) \cos(2\pi f + 1) \theta \int_{0}^{\frac{1}{2n+1}} \sin(2\pi f + 1) y_{n} d\theta \]  
(7.20)

The expression for the output of Fig. 7.8 is nothing but the Fourier-series expansion of the saw-tooth wave with a periodicity of 2m. Aside the common types of phase detection, there is another variety which is sometimes used for the so-called 'tunelock' loop. The phase detector characteristic is given by

\[ v_{d} = \frac{(1 - x) \sin \phi}{1 + x \cos \phi} \]  
(7.21)

The above type of phase characteristics can be realized with the help of the analog circuit as shown in Fig. 7.9. For further details, see
Fig. 7.9. The block diagram of a tuned-lock phase detector.

readers are referred to the literature [1-4].

The different types of phase detectors modify the phase locked loop responses in various ways. Later we will discuss them briefly.

7.2.2 VOLTAGE CONTROLLED OSCILLATORS

Various circuits can be designed to serve the basic purpose of the voltage controlled oscillator, i.e., to control the instantaneous frequency of an oscillator with the application of a voltage. But to fulfill the purpose of a phase-locked loop, a voltage controlled oscillator is required to satisfy the following criteria of goodness [1, 5]:

(1) Linearity of the variation of frequency with the control voltage; even a tolerance of 5% or 10% is not acceptable for the use of a PLL as a frequency discriminator.

(2) Reasonably large modulation sensitivity, i.e., the change of frequency with the control voltage.

(3) Adequate frequency stability. This is contradictory to the requirement of large modulation sensitivity.

(4) Large modulation bandwidth. This is needed for the use of PLL as a demodulator. This is again in direct opposition to the requirement of stability.

(5) Spectral purity of the output waveform. Voltage controlled oscillators that are commonly used are of the following types [1, 13, 14, 15]:

1) Astable multivibrators

The ?CO'S using astable multivibrators or relaxation oscillators, are useful up to a few megahertz. Here frequency variation is ob-
tained by means of the capacitor charging current. Circuit components should be chosen so that the stable operating as well as low noise generation is guaranteed. The transistors should be high speed devices with low leakage and low noise generation. Temperature difference between various circuit components should be reduced by providing heat sinks so as to minimize the frequency fluctuation due to temperature variation. To achieve better temperature stabilization of oscillation frequency sometimes a collector limit circuit is incorporated. Note that in this type of VCO's although the stability is poor yet the tuning range is outstanding and the linearity of frequency variation versus the modulating voltage is excellent. Moreover, the cost is also low.

2) **L-C oscillators**

For use up to a few hundred megahertz LC oscillators, using standard Hartley, Colpits and Clapp circuits, are designed. Here the frequency variation is obtained by means of a varactor, although saturable inductors have been used in some cases. At higher band of frequencies (microwave and millimeter wave) varactor tuned Gunn and IMPATT oscillators are used. However, at microwave frequencies, bias tuning is also preferred. In the age of vacuum tubes, resonant tubes were employed to obtain frequency modulation of the oscillator. Here the tuning range as well as the linearity of frequency variation with the control voltage is quite large but the stability factor is low.

3) **Crystal controlled oscillators**

Where high degree of stability is demanded, voltage controlled crystal oscillators are commonly used. It is known that the series resonant frequency of a quartz crystal can be varied to some extent. To achieve this small variation, a varactor diode is connected in series with crystal. This arrangement gives a certain amount of frequency variation without affecting the stability of the oscillator too much. For reasons of stability, usually AT-cut crystals operating in the third and fifth overtone mode are used. Temperature dependence of frequency of oscillation of an AT-cut crystal is cable in form with the temperature excursion. In view of this, the crystal is housed in a temperature controlled oven so as to improve the stability of the oscillator. Moreover, with the help of compensation
techniques frequency variations of less than $5 \times 10^{-6}$ from $-40^\circ\text{C}$ to $+60^\circ\text{C}$ can be achieved. At this point, it may be noted that the phase stability of the oscillator also depends on the mechanical stability of the crystal. Vibration sensitivity ($\Delta f/\Delta F$) of quartz itself is of the order of $2 \times 10^{-6}$ per g. Crystal vibration really affects the short term stability of the oscillator. Ageing also affects the long term stability of the oscillator and it ranges from $10^{-6}$ per year to $10^{-8}$ per year depending upon whether the crystal is solder-seal rugged-mount or it is a ribbon-mount cold-weld unit for 2.5 or 5.0 MHz frequency standards.

The specifications generally available for VCO's are shown in Table 7.1.

The specifications shown in Table 7.1 are given in terms of the centre frequency of the oscillators. For further information the interested readers may refer to the literature [7-13].

2.2.3 LOW-PASS FILTER

The output of the phase detector is connected to the voltage controlled oscillator through a low-pass filter. It modifies the response characteristics of the phase locked loop to a large extent. We shall consider the nature of modification in the next few chapters. Normally two types of low-pass filters are used, viz., passive and active forms. A common form of the low pass filter is shown in Fig. 7.10.

![Diagram of a low-pass filter](image)

Fig. 7.10. (a) The passive and (b) the active low-pass filters.

The transfer function of the low-pass filters are given by

$$\tilde{F}(j\omega) = \frac{1 + j\omega CR_2}{1 + j\omega C(R_1 + R_2)} = \frac{1 + j\omega T_2}{1 + j\omega T_1}$$  \hspace{1cm} (7.22)

where

$$T_1 = (R_1 + R_2)$$  \hspace{1cm} \text{and} \hspace{1cm} T_2 = CR_2$$
<table>
<thead>
<tr>
<th>Specifications</th>
<th>Multivibrator VCO</th>
<th>Varactor controlled VCO</th>
<th>Voltage controlled S-band VCO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency Deviation</td>
<td>±0.5% (0°C - 50°C)</td>
<td>±1 to ±4%</td>
<td>±0.001 to ±0.1%</td>
</tr>
<tr>
<td>Frequency Stability</td>
<td>±0.05% (0°C - 50°C)</td>
<td>±0.05% (0°C)</td>
<td>±0.00004 to ±0.1% (0°C - 50°C)</td>
</tr>
<tr>
<td>Linearity</td>
<td>±0.1%</td>
<td>±0.5%</td>
<td>±5 to 10%</td>
</tr>
<tr>
<td>Sensitivity</td>
<td>5 to 20% per volt</td>
<td>1 to 5% per volt</td>
<td>0.01% to 0.3% per volt</td>
</tr>
</tbody>
</table>
Phase Lock Theories and Applications

\[ F(j\omega) = \frac{1 + j\omega C R_4 A}{j\omega C R_3 + 1 + (1 - A) j\omega C R_4} = \frac{1 + j\omega T}{1 + j\omega T} \]  

(7.23)

However, for large values of the gain \( A \) (7.23) reduces to

\[ F(j\omega) = \frac{1 + j\omega T}{j\omega T} \]  

(7.24)

If the complex frequency variable \( s \) is used, the transfer functions of the filter are written as

\[ F(s) = \frac{1 + s T}{s T} \]  

(7.25)

and

\[ F(s) = \frac{1 + s T}{s T} \]  

(7.26)

In the operational notation the above filter transfer functions may be written as

\[ F(p) = \frac{1 + p T}{1 + p T} \]  

(7.27)

and

\[ F(p) = \frac{1 + p T}{p T} \]  

(7.28)

where \( p = \frac{d}{dt} \) is the Heaviside's operator. The purpose of writing the filter transfer function in the operational notation is to write the differential equations governing the behaviour of the phase locked loop in a compact form. For example, if the input to the linear filter with the transfer function \( F(p) \) is \( v(t) \) then the output is written as \( v(t) = F(p) v(t) \) which actually means the following

\[ v(t) = \frac{1 + T}{T} \frac{d}{dt} v(t) \]  

(7.29)

or

\[ v(t) = \frac{1 + T}{T} \frac{d}{dt} v(t) \]  

(7.30)
Further, it may be stated that the filter transfer function, expressed either in the Laplace transform notation or Fourier transform notation, may be represented as
\[ F(s) = F(p)_{p=0} \]
or
\[ F(j\omega) = F(p)_{p=j\omega} \]

### 7.3 Mathematical Analysis

Let us refer to the block diagrammatic representation of the phase-locked loop and try to express the operation of the loop in terms of analytical expressions. Let \( v_d(t) \) denote the output of the phase detector. Therefore, the output of the low-pass filter can be written as
\[ v_d(t) = \int_0^t v_p(u) f(t - u) \, du \]

Note the \( f(t) \) denotes the impulse response of the filter network. Assume that the input signal is of the form \( \sqrt{2}A \sin (\omega_d t + \Psi_1) \). Take the output of the voltage controlled oscillator as \( \sqrt{2}K_{v} \cos \omega_d t \) when the output of the phase detector is not connected to the VCO. Let \( \sqrt{2}K_{v} \cos (\omega_d t + \Psi_1 f(t)) \) represent the output of the phase detector when the output of the phase detector is connected to the VCO. If \( K_{v} \) is the sensitivity of the VCO in radians per volt, then by definition
\[
\frac{d\Psi}{dt} = K_{v}v_d(t)
\]
The negative sign of the filter output is absorbed in \( K_{v} \). That is the frequency of the VCO versus the control voltage has a negative slope if the filter output is negative and it is positive when the filter output is positive. Note that the phase difference between the output of the VCO and the reference input is
\[ \Psi = \omega_d t + \Psi_1 - \omega_p t - \Psi_1 \]  
(7.31)
Therefore, from (7.30) and (7.31) one writes
\[
\frac{d\Psi}{dt} = \omega_d - \omega_p + K_{v}v_d(t) + \frac{dv_d}{dt}
\]
(7.32)
Now for a sinusoidal phase detector, which is commonly used,
\[
v_d(t) = K_{v}K_{p}A \sin \Psi(t)
\]
(7.33)
and thus
\[ v_d(t) = AK_pK_i \int \sin \varphi(u)f(t-u) \, du \]  
(7.34)

Here \( K_p \) represents the gain of the phase detector. \( AK_pK_i \) is sometimes called the sensitivity of the phase detector and has the unit of volt per radian. If \( \varphi \) is small, one finds that \( AK_pK_i = v_d(t)/\varphi \).

In practical situations, the phase detector may incorporate limiters and amplifiers in which case the phase detector output is written as \( a \sin \varphi \). Here \( a \) stands for \( AK_pK_i \) and is called the phase detector sensitivity. If the filter network is absent, then
\[ v_d(t) = v_d(0) \]  
(7.35)

Thus (7.33) reduces to
\[ \frac{dv}{dt} = (a_1 - a_0) - AK_pK_i \sin \varphi + \frac{dv}{dt} \]  
(7.36)

or if
\[ \frac{dv}{dt} = 0, \]

then
\[ \frac{dv}{dt} = a_1 - a_0 - AK_pK_i \sin \varphi \]  
(7.37)

Comparing (7.37), the governing system equation of the PLL, with the phase equation of the injection synchronized oscillator (3.57), one finds that the loop equation of the PLL does not depend on the output amplitude of the local oscillator and the Q-value of the tuned circuit. This means that the band of synchronization of the PLL can be changed without affecting the stability of the oscillator itself. There are other points of advantage, which we will see later.

7.3.1 First Order Loop (16, 17, 13)
The equation (7.37) is a first order differential equation (nonlinear) and as such, in conformity with the terminology of servomechanism, the phase locked loop without the low pass filter is called a first order loop. Note further that the equation (7.37) gives the instantaneous beat angular frequency of the loop.

Now it is known that there are two solutions possible depending upon the relation between the open loop frequency error \( (a_1 - a_0) \)
and open loop gain $AK_3K_1K_r$. For example, when $(0_1 - 0_2) < AK_3K_1K_r$, the solution is $\psi = K_3K_1K_r$.

\[
\psi = 2 \arctan \left( \frac{1 - x}{x} \right) - \frac{AK}{2\sqrt{1 - x^2}} \sqrt{(1 - x^2)(1 + x^2)}
\] (7.38)

and when $(0_1 - 0_2) > AK_3K_1K_r$, the solution is given by

\[
\psi = 2 \arctan \left( \frac{\sqrt{1 + x^2} - 1}{x} \right) - \frac{AK}{2\sqrt{1 + x^2}} \sqrt{2x^2 - 1} (1 + x^2)
\] (7.39)

where $t_0$ is a constant of integration and $x$ is given by

\[
x = \frac{\theta_1 - \theta_0}{AK}
\] (7.40)

The reaction (7.38) indicates that the phase difference $\psi$ attains a steady state value in the long run. The reaction (7.39) indicates that the phase difference does not reach a constant value with time but varies periodically with time. In the former case when the phase difference attains a constant value, it indicated a zero value of the best angular frequency. The frequency of the local oscillator, therefore, becomes identical with the frequency of the synchronizing signal. In this case, we say that the local oscillator is locked or synchronized to the reference signal. The value of the steady state phase difference is given by

\[
\psi = \arcsin \left( \frac{\theta_1 - \theta_0}{AK} \right)
\] (7.41)

and the time taken to attain the locked condition is given by (c.f. 7.37)

\[
T_{\psi} = \int_{0}^{\psi} \frac{d\psi}{\theta_1 - \theta_0 - AK \sin \psi}
\] (7.42)

where $\theta_0$ is some initial phase.

Obviously, the time of acquisition as given by (7.42) will be infinite. In order to get a meaningful measure of the acquisition time, the upper limit of the integration is taken to be 30 per cent of the final steady state phase error $\arcsin (\theta_1 - \theta_0)/AK$. In this case, the time of acquisition becomes
\[ T = \arg \left( \frac{\cos \left( \frac{1}{2} \left( y + v \right) \frac{AK}{\cosh \left( \frac{1}{2} \left( y - v \right) \right)} \right)}{\sin \left( \frac{1}{2} \left( y + v \right) \frac{AK}{\cosh \left( \frac{1}{2} \left( y - v \right) \right)} \right)} \right) \]  

where 
\[ v = \arcsin \left( \frac{-1}{2} \frac{AK}{y} \right) \]  
and 
\[ \Omega = \alpha - v \]

The maximum value of the open loop frequency error up to which the loop can be locked is to the external synchronising signal is \( s_f \), which is called the locking range of the first order loop, and is called the hold-in range of the phase locked loop.

Let us now look into the behaviour of the loop when the open loop frequency error \( \Delta f \) is greater than the hold-in range of the loop \( s_f \). In this case we have already seen that the loop will not lock to the synchronising signal and so differentiate \( s_f \) with respect to time \( t \) and utilise the following identity

\[ \cos \left( \frac{1}{2} y \right) = 1 + \tan^2 \left( \frac{1}{2} y \right) \]

(7.44)

to write the following expression [4] for the instantaneous frequency of the loop

\[ f(t) = \frac{s_f(t)}{2} + \frac{\Delta f(t)}{2} \]  

(7.45)

where 
\[ 2f = AK \sqrt{2 - \frac{1}{2}} \left( f + i \right) \]

and 
\[ \Omega = \arcsin \sqrt{2 - \frac{1}{2}} \]

(7.46)

New putting
\[ r = x - \sqrt{2 - \frac{1}{2}} \]

(7.47)

and using the following relations

\[ 1 + \tau y = 1 + \frac{y}{\alpha \cos \theta} \]

(7.48)

Real part of
\[ \frac{1}{1 + \tau y} = 1 + \frac{y}{\alpha \cos \theta} \]

(7.49)

and
\[ \nu = 2f - 2 \Omega \]

one can easily show that
(7.45, 7.48, 7.49)
\[
\frac{\partial \theta}{\partial t} = \frac{AK}{\sqrt{2} - 1} + 2AK \sqrt{2}^{-1} \sum_{n=1}^{\infty} (-1)^n \cos \pi (\lambda - \beta)
\]

(7.59)

Therefore, the average beat frequency \(\langle \frac{\partial \theta}{\partial t} \rangle\) is the beat angular frequency given by

\[
\omega_b = \frac{\partial \theta}{\partial t} = \frac{AK}{\sqrt{2} - 1} - \sqrt{\left(\omega_0 - \omega_b\right)^2 - \left(\Delta K\right)^2}
\]

(7.51)

Hence, the average value of the VCO's frequency in the presence of the synchronizing signal is given by (cf. 7.31)

\[
\omega_{av} = \omega_0 + \frac{\partial \theta}{\partial t} = \omega_0 + \frac{\partial \theta}{\partial t} - \sqrt{\left(\omega_0 - \omega_b\right)^2 - \left(\Delta K\right)^2}
\]

(7.58)

If \(\Psi(t)\) is a periodic modulator with a mean zero, then the average value of the VCO's frequency is (cf. 7.51, 7.52)

\[
\omega_{av} = \omega_0 + \int (\omega_0 - \omega_b) - \sqrt{\left(\omega_0 - \omega_b\right)^2 - \left(\Delta K\right)^2}
\]

Referring to the relation (7.51) one finds that when the value of \(\omega_0 - \omega_b\) is greater than \(\Delta K\), the beat angular frequency \(\omega_b\) continuously decreases as \(\omega_0\) is brought closer to \(\omega_b\). It is interesting to note that when the magnitude of \(\omega_0 - \omega_b\) is large compared to \(\Delta K\), the beat angular frequency decreases linearly with \(\omega_0\). But, when \(\omega_0 - \omega_b\) becomes closer to \(\omega_b\), the drop in the value of the beat angular frequency with change in \(\omega_0\) becomes steeper. And when the difference \(\omega_0 - \omega_b\) becomes equal to \(\Delta K\), the beat angular frequency becomes zero. This means that the frequency of the local oscillator becomes equal to that of the synchronizing signal. This is what is meant by locking of the local oscillator to the synchronizing signal. A nature of variation of the beat angular frequency is shown in Fig. 7.11.

Refer to the relation (7.51) and observe that the average frequency of the local oscillator (VCO), under the unlocked condition, is not equal to the free-running frequency of the oscillator. If \(\Delta K\) is smaller but \(\omega_0 - \omega_b\) is larger or greater than thefree-running frequency depending upon whether the frequency of the forcing signal is less or greater than the free-running frequency of the local oscillator. The relation (7.51) further shows that the frequency of the VCO is shifted toward the frequency of the forcing signal. Thus we say that the frequency of the local
oscillator is 'pulled' under the influence of the synchronizing signal. The output of the phase detector is given by

\[ v_p = A \sin \varphi \quad \text{when } \omega_1 - \omega_2 < AK \]  
\[ v_p = A \sin \varphi \quad \text{when } \omega_1 - \omega_2 > AK \]  
\[ v_p = A \left( \frac{\sqrt{\frac{2}{3}} - 1}{x + \cos (2\varphi - \beta_0)} \right) \] (7.56)

That is, under unlocked condition, the relation as given by (7.55) can be written as (cf. 7.37, 7.42 and 7.50)

\[ v_p = A \left( 1 - \sqrt{\frac{2}{3}} \right) + A \frac{\sqrt{\frac{2}{3}} - 1}{x} \frac{(-1)^{n} \cos (2\varphi - \beta_0)}{(2x - \beta_0)} \] (7.56)

The variation of the phase detector output voltage with time, taking the detuning as a parameter, is shown in Fig. 7.12. Note that even in the unlocked state, the phase detector produces a d.c. voltage, which is responsible for pulling the oscillator frequency towards that of the synchronizing signal.

7.3.2 STABILITY OF THE FIRST ORDER LOOP

Referring to the equation (7.37) one finds that if \( AK_n^0 \beta \) is less than \( \omega_1 - \omega_0 \) the phase difference does not attain a steady state value, whereas if \( AK_n^0 \beta \) is greater than \( \omega_1 - \omega_0 \) then the phase difference attains a steady state value, which is given by
Fig. 7.12. The phase detector characteristics under unlocked conditions: (a) positive and (b) negative detuning.

\[ \frac{\Omega}{k} < -1 \]

That is

\[ \sin \varphi = \frac{\omega_0 - \omega_k}{A K_1 K_2 K_3} \] (7.57)

\[ \varphi = (2n + 1) \pi - \sin^{-1} \left( \frac{\omega_0 - \omega_k}{A K_1 K_2 K_3} \right) \] (7.58)
\[
\theta_0 = 2\pi n + \sin^{-1}\left(\frac{\omega_1 - \omega_0}{AK}\right)
\] (7.59)

This is shown in Fig. 7.13. It is now to be noted that all the values of the steady-state phase difference are not stable. By stability of an operating point we mean that if the system, by chance, moves/shifts a little from the operating point it will again come back to its initial operating point. On the other hand, an operating point is an unstable one, if the system, on moving away from the operating point, does not come back again to the initial operating point. The stable operating points can be found in the following way. Let us suppose that we deliberately change the phase difference by a very small amount \(\delta\) from the steady-state value \(\theta_0\). Therefore, the system equation under this condition can be written as

\[
\frac{d(\theta_0 + \delta)}{dt} = (\omega_1 - \omega_0) - AK \sin (\theta_0 + \delta)
\]

or

\[
\frac{dx}{dt} = -AK \cos \theta_0 \cdot x
\]

i.e.,

\[x = \text{const. exp}(-AK \cos \theta_0 \cdot t)\] (7.60)

where

\[K_\theta K_x K_y = K\] (7.61)

This indicates that if \(\cos \theta_0\) is positive, \(x\) will die out. This means that the system will be brought back to the initial state. Note that \(\cos \theta_0\) can take positive values provided \(\theta_0\) satisfies (7.59). On the other hand, if \(\theta_0\) satisfies (7.58), \(x\) will go on increasing. Thus the values of \(\theta_0\) corresponding to (7.58) give the unstable operating points. These are shown in Fig. 7.13.

The points \(A\) and \(B\) indicate the stable points whereas the points \(B\) and \(D\) are unstable points. However, if \((\omega_1 - \omega_0)\) is negative, the corresponding operating points of the phase-locked loop are \(E, F, G\) and \(H\). Out of these possible operating points, the points \(G\) and \(F\) are stable whereas the points \(F\) and \(H\) are unstable. These are also shown in Fig. 7.13.
1.4 Design Example

Design a phase-locked loop, operating at 120 MHz, so as to have a single-sided locking range of 10 KHz with received signal power of -90 dBm. It is required that the VCO frequency stability should be such that the maximum phase error excursion does not exceed 0.08 radians. Use a multiplier type phase detector.

If the carrier power is $P_c$, the locking range expression is

$$\sqrt{P_c} K_c K_p = 2\pi \times 10^4$$

since $P_c = 10^{-4}$ Watts, one finds

$$K_c K_p K_p = 2\pi \times 10^4$$

Let $\Delta f$ be the maximum frequency drift corresponding to the phase error excursion of 0.08 radians. Then one writes

$$0.08 = \arcsin \frac{2\pi \Delta f}{2\pi \times 10^4}$$

hence

$$\Delta f = 0.8 \times 10^3$$

Thus the frequency stability of the VCO required is
It is known that this value of frequency stability can only be realized with the help of a voltage controlled crystal oscillator (VCXO) (cf. Table 7.1). Further referring to the Table 7.1 one finds that the maximum frequency sensitivity of the VCXO, operating at 120 MHz, can be \(0.003 \times 120 \times 10^9\), i.e., 360 KHz/volt. Let us suppose that the VCXO has a sensitivity of 200 KHz/volt. Therefore, the required phase detector gain is

\[
K_p = \frac{10^6}{2X_1}
\]

Note that \(\sqrt{2K_p}\) is the rms amplitude of the VCO. Normally VCXO's, operating up to 20 MHz, are available. Therefore, the VCXO should be followed by a frequency multiplier of order 6. Thus the actual sensitivity of the VCXO is 200/6 KHz/volt. Note further that the frequency variation range of this VCXO is 10/6 KHz, which is permissible, because the maximum frequency deviation for a VCXO is 0.001 \(\times 20 \times 10^9\), i.e., 20 KHz (see Table 7.1).

REFERENCES

Chapter 8

Noise-Free Analysis of Linearised Loops

The response of a phase-locked loop to various types of signals will be studied in this chapter, when the tracking phase error is small. In such a situation, the periodic phase error characteristics of phase detectors, viz., sinusoidal, triangular, saw-tooth, etc., will be replaced by linear phase error characteristics. Linear PLL-theory is presented here for two reasons, namely, (1) linear PLL-theory is widely used in the design of phase locked tracking and communication systems and (2) a small phase error is usually desired for good tracking performances.

8.1 Linearised Loop Equation

Now referring to equations (7.32) through (7.34) one finds that the general equation that governs the operation of a PLL is a nonlinear differential equation. The nonlinearity is a periodic function of the phase error. The exact solution of this equation is difficult to obtain. We will discuss them later. However, when the voltage-controlled oscillator is closely following the reference input, the phase error between the two signals is obviously small, and as such the nonlinear periodic characteristics of the sinusoidal, triangular, and sawtooth phase detectors can be replaced by a linear relation. Thus, in this case, the loop equation can be written as

$$\frac{dy}{dt} = \omega_1 - \omega_y - 4K \int_0^t q(u) f(t-u) \, du + \frac{dy}{dt}$$

(8.1)

We will now write the filter response in the operational notation as

$$R(p) = \frac{1 + pT_s}{pT_s}$$

(8.2)
\[ f(p) = \frac{1 + \rho T_s p}{1 + \frac{\rho}{\alpha}} \]  
(8.3)

Now if one takes the filter of (8.2), one gets

\[ T_1 \frac{dx}{dt} = \nu + T_2 \frac{dx}{dt} \]  
(8.4)

and for the filter of (8.3) one finds

\[ T_1 \frac{dx}{dt} + \eta = \nu + T_2 \frac{dx}{dt} \]  
(8.5)

Recall that \( \eta \) is the output of the filter network with \( \nu \) as the input. Therefore, inserting (8.4) and (8.5) in (7.33) one gets (\( y_2 = AK_1K_2 \)) the following equations for the two types of filter (cf. 8.2, 8.3)

\[ T_1 \frac{dy_2}{dt} = -K_1 \left( \nu + T_2 \frac{dx}{dt} \right) + T_4 \frac{dy_0}{dt} \]  
(8.6)

i.e.,

\[ T_1 \frac{dy_2}{dt} + AK_1T_2 \frac{dx}{dt} + AK_2 = T_4 \frac{dy_0}{dt} \]  
(8.7)

The stability of the phase locked loops, incorporating the above two types of filter networks, can be easily studied with the help of (8.6) and (8.7) after writing the incremental equation as

\[ -T_1 \frac{dy_2}{dt} + AK_1T_2 \frac{dx}{dt} + AK_2 = 0 \]  
(8.8)

and

\[ T_1 \frac{dy_2}{dt} + (1 + AK_1T_2) \frac{dx}{dt} + AK_2 = 0 \]  
(8.9)

where

\[ \delta = \nu_2 + x \]  
(8.10)

Note that the differential equations (8.8) and (8.9) give the nature of variation of the disturbance \( x \) with time, \( x \) being an increment in the value of the phase error from its steady state value \( y \). The characteristic equation corresponding to (8.8) and (8.9) is of the form

\[ a_3 \dot{x} + a_2 \dot{x} + a_1 x = 0 \]  
(8.11)
In either of the cases, the solution may be written as
\[ x = A \exp (p_1 t) + B \exp (p_2 t) \]  
(8.12)
where \(A\) and \(B\) are constants and \(p_1\) and \(p_2\) are the roots of the characteristic equation. One finds that
\[
2p_1, 2p_2 = \frac{-A K_T}{T_4} \pm \frac{(A K_T^2 - 4 A K T_4)^{1/2}}{T_4}
\]  
(8.13a)
or
\[
2p_1, 2p_2 = \frac{(1 + A K T_4)}{T_4} \pm \frac{(1 + A K T_4^2 - 4 A K T_4)^{1/2}}{T_4}
\]  
(8.13b)
Thus in both the cases it is seen that the disturbance \(x\) dies out, indicating that the system returns to its initial state. This means that the system is stable. Referring to (8.13a) and (8.13b) one finds that the roots of the characteristic equation have always real negative parts. This indicates that the increment \(x\) will ultimately die out. In general, we can state that a system is stable if the real parts of all the roots of the characteristic equation are negative. Thus the second order phase locked loops using the above types of filters are stable.

8.2 Loop Transfer Functions and Root Locus Plots

Let us consider that an angle-modulated signal of the form \(\sqrt{2} A \sin (\omega_c t + \Psi(t))\) is fed to the input of a PLL, the centre frequency of the VCO being timed to the carrier frequency of the input signal. That is, the output of the VCO may be taken as \(\sqrt{2} K_v \cos (\omega_c t + \Psi(t))\). Note that \(\sqrt{2} K_v\) is the rms amplitude of the VCO output. Therefore, the phase detector output is given by
\[
\tau_p = AK_v K_c \sin (\Psi(t) - \Psi(t))
\]  
(8.14)
When the VCO is following the input signal closely, one may assume that the phase error is small and as such the relation (8.14) may be approximated as
\[
\tau_p = AK_v K_c (\Psi(t) - \Psi(t))
\]  
(8.15)
Hence one writes the following equation for the phase modulation of the VCO
\[
\frac{d^2 \theta}{dt^2} = AK_v K_c \int (\Psi(t) - \Psi(t)) \, dt
\]  
(8.16)
In either of the cases, the solution may be written as

$$x = A \exp(p_1 t) + B \exp(p_2 t)$$  \hspace{1cm} (8.12)

where $A$ and $B$ are constants and $p_1$ and $p_2$ are the roots of the characteristic equation. One finds that

$$2p_1, 2p_2 = -\frac{AKT_f}{T_1} \pm \left(\frac{\Delta K T_f}{T_1} - 4AKT_f\right)^{1/2}$$  \hspace{1cm} (8.13a)

or

$$2p_1, 2p_2 = -\left(1 + \frac{AKT_f}{T_1}\right) \pm \left(1 + \frac{AKT_f}{T_1} - 4AKT_f\right)^{1/2}$$  \hspace{1cm} (8.13b)

Thus in both the cases it is seen that the disturbance $x$ dies out, indicating that the system returns to its initial state. This means that the system is stable. Referring to (8.13a) and (8.13b) one finds that the roots of the characteristic equation have always real negative parts. This indicates that the increment $x$ will ultimately die out. In general, we can state that a system is stable if the real parts of all the roots of the characteristic equation are negative. Thus the second order phase locked loops using the above types of filters are stable.

### 8.2 Loop Transfer Functions and Root Locus Plots

Let us consider that an angle-modulated signal of the form

$$\sqrt{2/4} \sin(\omega_0 t + \Psi_f(t))$$

is fed to the input of a PLL, the centre frequency of the VCO being tuned to the carrier frequency of the input signal. That is, the output of the VCO may be taken as

$$\sqrt{2/4} \cos(\omega_0 t + \Psi_f(t))$$

Note that $\sqrt{2/4}$ is the rms amplitude of the VCO output. Therefore, the phase detector output is given by

$$r_p = AK_xK_s (\Psi_f(t) - \Psi_d(t))$$  \hspace{1cm} (8.14)

When the VCO is following the input signal closely, one may assume that the phase error is small and as such the relation (8.14) may be approximated as

$$r_p = AK_xK_s (\Psi_f(t) - \Psi_d(t))$$  \hspace{1cm} (8.15)

Hence one writes the following equation for the phase modulation of the VCO

$$\frac{d\Psi_f}{dt} = AK_xK_s \left(\Psi_f(t) - \Psi_d(t)\right) f(t - u) \, du$$  \hspace{1cm} (8.16)
Taking the Laplace transform of (3.16), one finds that the closed loop transfer function of the PLL is given by

\[ H(s) = \frac{\frac{\psi(s)}{\psi(s)}}{1 + AKF(s)} \]  
(8.17)

and similarly the open loop transfer function of the loop is given by

\[ G(s) = \frac{\frac{\psi(s)}{\psi(s)}}{1 + AKF(s)} = \frac{AKF(s)}{s} \]  
(8.18)

Here, \( F(s) \) again denotes the transfer function of the loop filter.

At this point let us introduce ourselves to certain nomenclature for classifying various types of PLLs. We will follow the usual convention. That is, the order of the loop is designated by the total number of poles of \( G(s) \), whereas the type of the loop is designated by the number of poles at the origin in \( G(s) \).

Now assuming that the filter transfer functions are of the forms

\[ F(s) = \frac{1 + sT_1}{sT_2} \]

and

\[ G(s) = \frac{1 + sT_1}{sT_2} \]

It is shown that the closed loop transfer functions of the PLL are given by

\[ H(s) = \frac{AK(1 + sT_1)}{sT_1 + AKT_2 + AK} \]  
(8.19)

and

\[ H(s) = \frac{AK(1 + sT_1)}{sT_1 + AKT_2 + AK} \]  
(8.20)

Thus, one finds that the denominator polynomials of (8.19) and (8.20) coincide with the characteristic equations of (8.5) and (8.9). Hence the roots of the characteristic equations of the system coincide with those of the denominator polynomials of the closed loop transfer functions. In other words, the roots of the characteristic equation are the poles of the closed loop transfer function. Hence for stability it is necessary that the transfer function should have no poles on the right half of the complex frequency plane. Referring to (8.19) and (8.20), one further finds that the location of the poles on the \( s \)-plane depends on the loop gain \( AK \). The locus of the poles, which is a straight line, is called the root locus plot. Following the standard proce-
dure as given in any textbook [1] on automatic control, root locus plots of some typical phase locked loops are shown in Fig. 8.1.

Fig. 8.1. The root-locus plots of (a) the first order PLL, (b) the PLL with a perfect integrating filter, (c) a PLL with an imperfect integrating filter, and (d) a third order PLL.

In the following we will consider root locus plots [2] of various types of phase locked loops.

Case I: \( F(s) = \frac{1}{s} \): First order, type-1 loop

\[
H(s) = \frac{AK}{s + AK}
\]

This indicates that \( H(s) \) has a simple real pole at \(-AK\) radians per second and a simple zero at infinity. The root locus plot is shown in Fig. 8.1a. Note that the location of the closed loop poles moves along the negative real axis with the increase of open loop gain \( AK \).

Case II: \( F(s) = 1 + \frac{s}{s_1} \): Second order, type-2 loop

\[
H(s) = \frac{AK(1 + s/s_1)}{s^2 + ASs_2 + AK}
\]

This indicates that \( H(s) \) has a pair of poles. Since the order of the
denominator polynomial is only second degree, one can easily write
an equation of the root loci by writing the real and imaginary parts
of the roots of \( s^2 T_T + s T_T + 1 = 0 \). Thus the equation of the
locus is easily seen to be

\[
\omega^2 + \left( \frac{\pi}{2} + \frac{1}{T_T} \right)^2 = \frac{1}{T_T^2},
\]

which describes a circle in the complex frequency plane with centre
at \(-1/T_T, 0\) and radius \( 1/T_T \). The root locus plot is shown in Fig.
8.1b. The roots become complex with negative real parts with the
increase of the open loop gain \( AK \). But for \( AK > 1/T_T \), the roots
become real again.

Case III: \( F(s) = \frac{1 + sT_T}{1 + sT_T} \) Second order, type 1-loop

\[ H(s) = sT_T + (1 + AK) s + AK \]

following exactly the method as outlined in the Case II, one can
easily show that the equation of the root locus is given by

\[
\omega^2 + \left( \pi - \frac{1}{\pi T_T} \right)^2 = \frac{1}{T_T^2} (1 - T_T/T_T)
\]

which indicates a circle of radius \( T_T \) \( 1/T_T \sqrt{1 - T_T/T_T} \) with centre at
\(-1/T_T, 0\). The root locus plot is shown in Fig. 8.4. The root locus
plot is similar to that of Fig. 8.1b, except that the roots do not be-
come complex as soon as \( AK \) becomes greater than zero, but the
complex roots appear when \( AK \) becomes greater than
\((2T_T - T_T) - 2\sqrt{T_T - T_T^2} T_T^2 \) and remains complex with negative
real parts till the value of \( AK \) is

\[
2T_T - T_T - 2\sqrt{T_T - T_T^2} T_T^2,
\]

When \( AK \) becomes infinitely large one of the real roots ends at \(-1/T_T \)
whereas the other merges to infinity.

Case IV: \( F(s) = \frac{(1 + sT_T)(1 + sT_T)}{sT_T T_T} \) Third order, type 3 loops.

\[ H(s) = \frac{AK(sT_T + sT_T + 1)}{sT_T T_T + AK T_T + sT_T + 1} \]

\[ \frac{AK}{sT_T T_T + AK T_T + sT_T + 1} \]
The root locus plot is shown in Fig. 8.1d. From the plot it is seen that as soon as the open loop gain $AK$ becomes greater than zero, one of the roots moves along the negative real axis whereas the other two roots become complex with positive real parts, and the real parts of this pair of roots remain positive till the value of $AK$ equals $\frac{T_1 T_2}{(T_1 T_2 + T_3 T_4)}$. This indicates that the loop is unstable for values of $AK$ less than $\frac{T_1 T_2}{(T_1 T_2 + T_3 T_4)}$. Thus for stability $AK$ must be greater than $\frac{T_1 T_2}{(T_1 T_2 + T_3 T_4)}$. Hence the loop is conditionally stable.

8.3 Transient Response

In this section we will discuss the transient response of the PLL to signals which suffer sudden changes in the phase or frequency.

Phase-locked loops are used to track communication signals emitted by moving space vehicles and as such the signals received at the ground station are subjected to Doppler shifts. Tracking Doppler frequency changes along with other measurements are used to establish the position and velocity of the vehicle. Tracking signals from space vehicles are important in two occasions, viz. (i) when the space vehicle is accelerating radially from the observer, and (ii) when the satellite is traversing its orbit and passing overhead. These situations are illustrated in Fig. 8.2. Let us consider Fig. 8.2a, which indicates that the space vehicle is accelerating radially from the stationary observer. The electric field strength of a sinusoidal wave arriving at the receiver can be written as

$$\frac{E}{E_0} = \text{Re} \left[ \frac{A}{E_0} \exp \left( j \phi(r, t) \right) \right]$$  (8.21)

Here $\phi(r, t)$ denotes the phase of the received signal. Therefore, the instantaneous frequency of the received wave can be written as

$$\nu = \frac{d\phi}{dt}$$  (8.22)

Let us suppose that the refractive index of the medium through which the space vehicle moves is \(n_0\). Then

$$\phi(r, t) = \omega_0 \left( t - \frac{r}{c} \right)$$  (8.23)

where $\omega_0$ is the transmitted frequency and $c$ is the velocity of light.

Thus
\[ \frac{d}{dt} \delta(t', t) = -\frac{c}{v} \frac{d}{dt}(v) \frac{dv}{dt} \]

where \( v \) is the velocity of the vehicle. Therefore, the Doppler shift is

\[ v(t) = -\frac{c}{v} \frac{d}{dt}(v) \cdot (t + \Delta t) \]

Now when the space vehicle is accelerating, one gets

\[ v = c \frac{\Delta v}{\Delta t} \]

Therefore, from (8.25) and (8.26) one gets

\[ v_{DP} = -\frac{c}{v} \left( \frac{d}{dt}(v) \cdot (t + \Delta t) \right) dt \]
The first term (dn/dt) gives the Doppler shift due to the variation of the refractive index of the medium, whereas, the second term gives the regular Doppler shift. The effect of the first term becomes almost insignificant when frequency of the radio wave becomes high. However, for precision range and velocity measurement, the influence of this term is to be taken into consideration. Referring to (8.27) one finds that the Doppler shift is proportional to time. This variation of the Doppler frequency shift with time is shown in Fig. 8.2c.

Let us consider Fig. 8.2a, which depicts a simplified path of the satellite when it flies far off from the ground station. Let us suppose that the space vehicle is moving in the $\mathbf{r}$-direction. Let $r_n$ be the radial distance of the satellite from the observer when it is passing overhead and $r$ denotes the radial distance of the satellite when it has moved away from the overhead position. Thus the received field strength can be written as

$$E = \text{Re} \left[ E_0 \exp \left( j \phi(r, \omega) \right) \right]$$

Therefore,

$$\omega_0 = d \phi(\mathbf{r}, \omega)/d\mathbf{r}$$

$$r^2 = r_n^2 + (\mathbf{r} \cdot \mathbf{r})$$

where $\mathbf{r}$ is the speed of the satellite. Thus Doppler shift is obtained from (8.25) as

$$\omega_0 = -\omega/c \left( \frac{dr}{dt} \cdot \mathbf{r} + \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} \right)$$

Therefore, from (8.31) and (8.29) one gets

$$\omega_0 \sim \frac{\omega}{c} \left( \frac{\mathbf{r} \cdot d\mathbf{r}}{d\mathbf{r} + v^2/2} \right)$$

when $t < 0$ (8.32)

and

$$\omega_0 \sim -\frac{\omega}{c} \left( \frac{\mathbf{r} \cdot d\mathbf{r}}{d\mathbf{r} + v^2/2} \right)$$

when $t > 0$ (8.33)

This Doppler shift is shown in Fig. 8.2.

Strictly speaking, the above result gives an approximate character of the Doppler shift. However, a detailed treatment taking into consideration of the variations of the optical path length, refractive index and others can be found in the references [1, 6].
Thus, in the above two situations we find that the instantaneous frequency of a received signal varies either slightly or otherwise with time. In order to be able to receive such signals one has to tune the receiver accordingly. But since the rate of Doppler shift is quite high, manual tuning is not possible. However, a phase-locked loop has the capability of achieving the said purpose. In many situations the frequency of the received signal suddenly changes. This happens in the case of frequency shift keying signals. In such cases the phase of the received signal may change suddenly. This occurs in the case of phase shift keying signals. It is concluded that if a phase-locked loop is to operate faithfully in these situations, one has to study the response of a PLL to signals undergoing various types of modulation, such as phase shift, frequency shift and frequency ramp.

In the following we will assume that the loop is initially in lock and has a null phase error. Further, we will assume that the loop will follow the signals with different modulations with small phase error. That is, the loop behavior can be studied with the linear system equations.

Thus here \( \omega_0 = \omega_0 + \Delta \) and \( \phi = \phi + \Delta \). Moreover if the loops incorporate a rate triangular or sawtooth phase detector, we assume that the zone of operation does not exceed the linear region. Therefore, the loop equation (8.35) can be written as:

\[
\frac{d\phi}{dt} + \Delta K \int_{0}^{t} \phi(t', \tau - \omega_0) d\omega_0 = \frac{\Delta \omega_0}{\Delta K} \tag{8.36}
\]

Now taking the Laplace transform we obtain:

\[
\Phi(s) = \frac{\mathcal{L}[\phi(t)]}{s + \Delta K \mathcal{L}[\omega_0]} \tag{8.36}
\]

Note that \( \phi' \) is a complex frequency variable.

### 8.3.1 Phase Step Response

Let us assume that at time \( t = 0 \) a phase step signal is applied to the input of a PLL. Therefore,

\[
\Phi(s) = \frac{W_0 s}{s + \Delta K \mathcal{L}[\omega_0]} \tag{8.36}
\]

where \( W_0 \) is the amplitude of the phase step. Again for a first order loop
Thus
\[ F(s) = 1 \]  
(8.37)

and
\[ \varphi(s) = \frac{\nu_s}{s + AK} \]  
(8.38)

The steady state value of the phase error \( \varphi \) can be found easily by applying the final value theorem, viz.,
\[ \varphi = \lim_{s \to 0} \varphi(s) = \lim_{s \to 0} s \cdot \nu(s) \]  
(8.39)

Therefore, from (8.38) and (8.39) one finds that
\[ \varphi = 0 \]  
(8.40)

Now taking the inverse transform of (8.38) one finds that
\[ \varphi(t) = \nu_s \exp(-AKt) \]  
(8.41)

For a second order loop using the perfect and imperfect integrators having respectively the transfer functions
\[ F(s) = \frac{1 + sT_2}{sT_2} \]  
(8.42)

and
\[ F(s) = \frac{1 + sT_3}{1 + sT_3} \]  
(8.43)

e one gets from (8.35)
\[ \nu_d(t) = \frac{sT_2 \nu_s}{T_2^2 + AKT_2^2 + AK} \]  
(8.44)

and
\[ \nu_p(t) = \frac{(1 + sT_3) \nu_s}{T_3^2 + (AKT_2 + 1)T_3^2 + AK} \]  
(8.45)

where \( \nu_d(t) \) and \( \nu_p(t) \) are phase errors corresponding to the loops having perfect and imperfect integrators respectively. By applying the final value theorem it is seen from (8.44) and (8.45) that the steady state phase errors are
\[ \nu_d = 0 \]  
(8.46)

and
\[ \nu_p = 0 \]  
(8.47)

Now putting
\[ c_1 = \frac{4K}{T_1} \]  
(8.48)

and
\[ 2\epsilon c_1 = \frac{AK}{T_1} \]  
(8.49)
\[ 2L_{\text{Lm}} = 1 + \frac{4AKF_a}{L_c} \]  
\[ (8.50) \]

One re-writes (8.44) and (8.45) as

\[ q_a(s) = \frac{v_P a}{s^2 + 2AKa + a^2} \]  
\[ (8.51) \]

and

\[ q_a(s) = \left( s + \frac{a^2}{4K} \right) v_P \alpha \]  
\[ s^2 + 2KL a + a^2 \]  
\[ (8.52) \]

or,

\[ q_a(s) = \left[ s + \alpha_a \left( \xi_1 + \sqrt{\xi_1^2 - 1} \right) \right] \left[ s + \alpha_a \left( \xi_1 - \sqrt{\xi_1^2 - 1} \right) \right] \]  
\[ (8.51a) \]

and

\[ q_a(s) = \left[ s + \alpha_a \left( \xi_1 + \sqrt{\xi_1^2 - 1} \right) \right] \left[ s + \alpha_a \left( \xi_1 - \sqrt{\xi_1^2 - 1} \right) \right] \]  
\[ (8.52a) \]

Therefore, taking the inverse Laplace transforms of (8.51a) and (8.52a) one gets, when \( \xi_1 \gg 1 \)

\[ q_a(t) = V_a \left[ \cosh \left( \alpha_a \sqrt{\xi_1^2 - 1} t \right) \right. \]  
\[ - \frac{\xi_1}{\sqrt{\xi_1^2 - 1}} \sinh \left( \alpha_a \sqrt{\xi_1^2 - 1} t \right) \left. \right] e^{-\alpha_a t} \]  
\[ (8.53) \]

when \( \xi_1 \gg 1 \)

\[ q_a(t) = V_a \left( 1 - \alpha_a t \right) e^{-\alpha_a t} \]  
\[ (8.54) \]

and when \( \xi_1 < 1 \)

\[ q_a(t) = V_a \left[ \cos \left( \alpha_a \sqrt{1 - \xi_1^2} t \right) \right. \]  
\[ - \frac{\xi_1}{\sqrt{1 - \xi_1^2}} \sin \left( \alpha_a \sqrt{1 - \xi_1^2} t \right) \left. \right] e^{-\alpha_a t} \]  
\[ (8.55) \]

Moreover, when \( \xi_1 \gg 1 \)
\[ \theta(t) = \Psi(t) \left[ \cosh (\alpha_{0} \sqrt{1-t} - t) \right. \\
+ \frac{\alpha_{0}}{\sqrt{1-t}} \left. \sinh (\sqrt{1-t} - \Omega_{0} t) \right] e^{-\alpha t} \] (8.56)

when \( \xi = 1 \),

\[ \theta(t) = \Psi(t) \left[ 1 + \frac{\alpha_{0}}{\sqrt{1-t}} \right] e^{-\alpha t} \] (8.57)

and when \( \xi < 1 \),

\[ \theta(t) = \Psi(t) \left[ \cos (\alpha_{0} \sqrt{1-t} - \Omega_{0} t) \right. \\
+ \frac{\alpha_{0}}{\sqrt{1-t}} \left. \sin (\alpha_{0} \sqrt{1-t} - \Omega_{0} t) \right] e^{-\alpha t} \] (8.58)

### 8.3.2 Frequency Step Input

For a frequency step input one has

\[ \Psi(t) = \frac{\Delta_{0}}{\Delta} \] (8.59)

where \( \Delta_{0} \) is the size of the frequency step.

Therefore, for a first order loop

\[ \phi(t) = \Delta_{0} \] (8.60)

From the final value theorem (cf. 8.39)

\[ \phi = \Delta_{0} \] (8.61)

Taking inverse Laplace transform of (8.60) one gets

\[ \phi(t) = \frac{\Delta_{0}}{\Delta} (1 - e^{-\Delta t}) \] (8.62)

For a second order loop with perfect integrator (cf. (8.42)) one gets

\[ \phi(t) = \frac{\Delta_{0}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \] (8.63)

Therefore, the steady state value is
\[ \varphi_\nu = 0 \]  
(8.64)

Now taking inverse transform of (8.63) one gets

when \( \xi_1 > 1 \),
\[ \varphi_\nu(t) = \frac{\Delta \alpha}{\omega_u} \sinh (\omega_u \sqrt{1 - \xi_1} t) e^{-\omega_u t} \]  
(8.64a)

when \( \xi_1 = 1 \),
\[ \varphi_\nu(t) = \frac{\Delta \alpha}{i \omega_u} \sin(i \omega_u t) e^{-i \omega_u t} \]  
(8.64b)

when \( \xi_1 < 1 \),
\[ \varphi_\nu(t) = \frac{\Delta \alpha}{\omega_u} \sin (\omega_u \sqrt{1 - \xi_1} t) e^{-\omega_u t} \]  
(8.64c)

By differentiating and putting \( \frac{d^2}{dt^2} \varphi_\nu = 0 \) in (8.64c) one finds the time at which maximum \( \varphi_\nu(t) \) occurs. That is,
\[ \tan (\omega_u \sqrt{1 - \xi_1} t) = \left( \frac{\sqrt{1 - \xi_1}}{\xi_1} \right) \]  
(8.65)

Therefore, the phase locking range is obtained by putting \( \varphi_\nu(t)_{\text{max}} = \pi/2 \), so that the loop does not slip cycle in following the step change in frequency. Thus
\[ \Delta \omega_{\text{max}} = \frac{\pi}{2} \frac{\omega_u}{\omega_u} \exp \left( -\frac{\xi_1}{\sqrt{1 - \xi_1}} \tan^{-1} \frac{\sqrt{1 - \xi_1}}{\xi_1} \right) \]  
(8.66)

Now for a second order loop with imperfect integrator (cf. (8.43)), one gets
\[ \varphi_\nu(t) = \frac{\Delta \alpha}{\sqrt{p^2 + 2 \Delta \alpha \omega_u + \omega_u^2}} + \frac{\Delta \alpha}{\sqrt{p^2 + 2 \Delta \alpha \omega_u + \omega_u^2}} \]  
(8.67)

Therefore, the steady state value of \( \varphi_\nu(t) \) is given by
\[ \varphi_\nu = \frac{\Delta \alpha}{\Delta K} \]  
(8.68)

Note that this is the same as that of a first order loop.
Taking the inverse Laplace transform of (8.67) one gets when $\xi > 1$,

$$
\varphi(t) = \frac{\Delta \alpha}{AK} + \frac{\Delta \alpha}{\alpha_0} \left( 1 - \frac{\xi \alpha_0}{AK} \right) \sinh \left( \alpha_0 \sqrt{1 - \xi^2} t \right) - \frac{\alpha_0}{AK} \cosh \left( \alpha_0 \sqrt{1 - \xi^2} t \right) \exp \left( - \xi \alpha_0 t \right)
$$

(8.69a)

when $\xi = 1$

$$
\varphi(t) = \frac{\Delta \alpha}{AK} + \frac{\Delta \alpha}{\alpha_0} \left( \alpha_0 - \frac{\alpha_0^2}{AK} \right) t - \frac{\alpha_0}{AK} \exp \left( - \alpha_0 t \right)
$$

(8.69b)

when $\xi < 1$

$$
\varphi(t) = \frac{\Delta \alpha}{AK} + \frac{\Delta \alpha}{\alpha_0} \left( 1 - \frac{\xi \alpha_0}{AK} \right) \sin \left( \alpha_0 \sqrt{1 - \xi^2} t \right) - \frac{\alpha_0}{AK} \cos \left( \alpha_0 \sqrt{1 - \xi^2} t \right) \exp \left( - \xi \alpha_0 t \right)
$$

(8.69c)

To obtain the maximum value of $\varphi(t)$, we differentiate (8.69c) and put it equal to zero to find the time at which maximum $\varphi(t)$ occurs.

$$
\alpha_0 \sqrt{1 - \xi^2} A_1 = \tan^{-1} \frac{\sqrt{1 - \xi^2}}{\xi} + \theta
$$

(8.70)

Hence

$$
(\varphi(t))_{\text{max}} = \frac{\Delta \alpha}{AK} + \frac{\Delta \alpha}{\alpha_0} \varphi_0 \sin \left( \alpha_0 \sqrt{1 - \xi^2} A_1 - \theta \right) \exp \left( - \xi \alpha_0 t \right)
$$

where

$$
\varphi_0 = \frac{(1 - \xi \alpha_0)AK}{\sqrt{1 - \xi^2}} + \frac{\alpha_0^2}{AK}
$$

$$
\theta = \arctan \frac{\alpha_0 \sqrt{1 - \xi^2}}{\xi AK - \xi \alpha_0}
$$

Therefore, the overshoot is given by

$$
\gamma = (\varphi(t))_{\text{max}} - \varphi_0
$$

$$
= \frac{AK}{\alpha_0} \varphi_0 \sqrt{1 - \xi^2} \exp \left( - \frac{\xi \alpha_0}{\sqrt{1 - \xi^2}} \tan^{-1} \frac{\sqrt{1 - \xi^2}}{\xi} - \theta \right)
$$

(8.71)
8.3.3 Frequency Ramp Input

In this case the change in input frequency is linearly proportional to time and as such one has

$$\varphi(t) = \frac{R}{\omega} t$$  \hspace{1cm} (8.72)

where $R$ is the slope of the input frequency variation of the input signal.

For a first order PLL,

$$\varphi(t) = \frac{R}{\omega} (t - e^{-\omega t})$$  \hspace{1cm} (8.73)

Hence,

$$\varphi(t) = \frac{R}{\omega} t - \frac{R}{\omega \omega K} (1 - e^{-\omega K t})$$  \hspace{1cm} (8.74)

For a PLL with a perfect integrator (cf. (8.42)) one has

$$\varphi(t) = \frac{R}{\omega \omega K} (s + \omega K \omega K + \omega K \omega K)$$  \hspace{1cm} (8.75)

The steady state phase error in this case (cf. (8.39)) is

$$\varphi_{ss} = \frac{R}{\omega \omega K}$$  \hspace{1cm} (8.76)

Taking the inverse Laplace transform of (8.73) one gets

when $\xi > 1$

$$\varphi(t) = \frac{R}{\omega \omega K} \frac{\cosh (\omega \sqrt{\xi^2 - 1}) - 1}{\sqrt{\xi^2 - 1}}$$

$$- \frac{\xi_0}{\sqrt{\xi^2 - 1}} \sinh (\omega \sqrt{\xi^2 - 1}) e^{-\omega \xi^2 t}$$  \hspace{1cm} (8.77)

when $\xi = 1$

$$\varphi(t) = \frac{R}{\omega \omega K} \frac{1 + \omega^2}{\omega} e^{-\omega t}$$  \hspace{1cm} (8.78)

when $\xi < 1$

$$\varphi(t) = \frac{R}{\omega \omega K} \frac{\cosh (\omega \sqrt{\xi - 1}) - 1}{\sqrt{\xi - 1}}$$

$$+ \frac{\xi_0}{\sqrt{1 - \xi^2}} \sin (\omega \sqrt{1 - \xi^2}) e^{-\omega \xi^2 t}$$  \hspace{1cm} (8.79)
Phase Lock Theories and Applications

Now for a PL with an imperfect integrator (cf. (8.43)) one gets

$$\psi(s) = \frac{R}{\delta(s^2 + 2\delta_s\omega_s + \omega_s^2)} + \frac{\omega_s^2}{AK}$$

(8.80)

From which one finds that the steady state phase error is unbounded, i.e.

$$\psi_\infty \to \infty$$

Taking the inverse Laplace transform of (8.80) one gets

when \( \xi_4 > 1 \)

$$\psi_4(t) = \frac{R}{\omega_s^2} \left( 1 - 2\delta_s\omega_s/\omega_s^2 + 1 \right) e^{-\omega_s^2 t}$$

(8.87)

when \( \xi_4 = 1 \)

$$\psi_4(t) = \frac{R}{\omega_s^2} \left( 1 - \frac{\omega_s^2}{AK} \right) + \frac{R}{AK} e^{-\omega_s^2 t}$$

(8.82)

when \( \xi_4 < 1 \)

$$\psi_4(t) = \frac{R}{\omega_s^2} \left( 1 - \frac{\omega_s^2}{AK} \right) e^{-\omega_s^2 t}$$

(8.83)
The response of the phase locked loop to different types of modulation is shown in Fig. 8.3 and Fig. 8.4. Referring to Fig. 8.3

One finds that the first order loop can stay in the locked condition for the cases of step changes in phase and frequency. However, the first order loop cannot remain in the locked condition for Doppler input signal. Referring to Fig. 8.4a one finds that a second order loop with a perfect integrator in the loop will not be pushed out of the locked condition for the Doppler rate input signal, but in following this type of signal the loop shows a certain finite phase error. However, a second order loop with an imperfect integrator cannot stay in lock for a frequency ramp modulated signal. Comparing the set of curves of Fig. 8.3 with those of Fig. 8.4 one finds that the incorporation of a filter network improves the performance of a
Fig. 8.4. Transient responses of a PLL with perfect integrating filter to the (a) phase-step, and (b) frequency step signals.

higher order loop over that of a first order loop in the following respects:

1) With proper adjustment of the damping factor, the loop res-
response to an abrupt change in the input phase or rate can be improved to a great extent.

2) With proper choice of filter network, the loop can be made to stay in lock for certain types of modulated signals which cannot throw a first order PLL out of the locked condition. However, to get a null phase error in the case of Doppler rate signal, one has to use a third order loop with a filter network of the form

\[
\left(1 + x T_3 \right) \left(1 + x T_4 \right) \frac{1}{s^2 T_5^2 + 1}.
\]

When this type of filter network is used, the loop becomes conditionally stable.

It is to be noted that the damping of the loop is to be adjusted in such a way so as to reduce the phase settling time to as low a value as possible. Moreover, the addition of a filter network also alters the operation of the loop in other respects, i.e., the pull-in and pull-out characteristics and noise filtering properly, which we will discuss later.

8.4. Response to an Angle Modulated Signal

Let us suppose that a single tone angle-modulated signal of the form \( \phi(t) \cdot \text{sin}(\omega t + \Psi_0(t)) \) is tracked by a PLL, the VCO output of which is of the form \( \omega(t) = \frac{2K_I \text{cos}(\omega t + \Psi_0(t))}{1 + 2xK/I} \). Further assume that the loop parameter is chosen in such a way that linear operation of the loop is guaranteed. Thus the phase error \( \phi(t) = \Psi_0(t) - \Psi_1(t) \) is given by (cf. 8.17)

\[
\phi(t) = \frac{1}{x} \frac{2K/I}{1 + 2xK/I} \Psi_0(t).
\]

For a single tone signal \( \Psi_0(t) \cdot \text{cos}(\omega t) \), phase-modulating a wave, we write

\[
\Psi_0(t) = \frac{2K/I}{1 + 2xK/I} \text{cos}(\omega t)
\]

and for a frequency modulated wave we write

\[
\Psi_1(t) = \frac{m_0}{4\omega_n} \sin \omega_n t.
\]

Therefore, for a first order loop \( \mid L(s) \mid = 1 \), one finds that for a
phase modulated wave, the steady state phase error is
\[
\varphi = \frac{\omega_m \theta_i}{\sqrt{\omega_m^2 + (AK)^2}} \cos \left( \omega_m t + \frac{\pi}{2} - \arctan \frac{\omega_m}{AK} \right) \tag{8.87}
\]
and for a frequency modulated wave
\[
\varphi = \frac{\omega_m \theta_i}{\sqrt{\omega_m^2 + (AK)^2}} \sin \left( \omega_m t + \frac{\pi}{2} - \arctan \frac{\omega_m}{AK} \right) \tag{8.88}
\]
It is to be noted that the loop parameters and \( b_i \) or \( m \) should be chosen in such a way that \(| \varphi | \) is less than \( \pi / 2 \) for our analysis to be valid. Referring to (8.88) one further finds that as \( \omega_m \) decreases, \( \varphi \) increases for a fixed value of the frequency deviation. As this happens, the bounds of \( \varphi \) may go beyond \( \pi / 2 \) and consequently distortion will appear at the output.

For a second order loop using the filter transfer functions of (8.42) and (8.43), it is easy to show that for a phase modulated wave
\[
\varphi(f, \omega_m) = \left( \frac{\omega_m^2}{\omega_m^2 + \frac{1}{2} L_i^2 m \omega_m + \omega_i^2} \right)^{\frac{1}{2}} \tag{8.89}
\]
and
\[
\varphi(f, \omega_m) = -\left( \frac{\omega_m^2}{\omega_m^2 + \frac{1}{2} L_i^2 m \omega_m + \omega_i^2} \right)^{\frac{1}{2}} \tag{8.90}
\]
and for a frequency modulated wave the expressions for \( \varphi(f, \omega_m) \) and \( \varphi(f, \omega_m) \) will be similar to those of (8.89) and (8.90) except that \( \theta_i \) 's to be replaced by \( \theta_i \) 's. Hence we write
\[
\varphi_1 = \frac{\omega_m \theta_i}{\sqrt{(\omega_m^2 - \frac{1}{2} L_i^2 m \omega_m + \omega_i^2)}} \cos \left( \omega_m t + \frac{\pi}{2} - \arctan \left( \frac{\omega_m}{\frac{1}{2} L_i^2 m \omega_m - \omega_i^2} \right) \right) \tag{8.91}
\]
and
\[
\varphi_2 = \frac{\omega_m \theta_i}{\sqrt{(\omega_m^2 - \frac{1}{2} L_i^2 m \omega_m + \omega_i^2)}} \cos \left( \omega_m t + \frac{\pi}{2} + \arctan \left( \frac{\omega_m}{\frac{1}{2} L_i^2 m \omega_m - \omega_i^2} \right) \right) \tag{8.92}
\]
Similar expressions for the angle modulated wave can be written with \( \phi \) replaced by \( m \) and \( \cos \) terms replaced by \( \sin \) terms. Plots of \( |q_1|/|m_1| \) or \( |q_1|/|m_2| \) and \( |q_2|/|m_2| \) or \( |q_2|/|m_1| \) are shown in Figs. 8.5a and 8.5b respectively. Note that the phase detector output

\[
\nu_p = AK_1 \nu_0 \frac{\cos \theta}{\sqrt{\tan^2 \theta + (AK)^2}} \sin \left( \theta_p - \arctan \frac{\tan \phi}{\nu_0} \right)
\]  

(8.93)

Similarly, the phase detector output for a FM wave is

\[
\nu_p = AK_1 \nu_0 \frac{2m_1 \Delta f}{\sqrt{\tan^2 \theta + (AK)^2}} \cos \left( \theta_p - \arctan \frac{\tan \phi}{\nu_0} \right)
\]  

(8.95)

Looking at the expression (8.94), one finds that a first order PLL acts like a phase demodulator followed by a high pass filter of time constant \( 1/AK \). Similarly, on examination of (8.95) it is seen that for a FM signal, a first order PLL behaves like an discriminator followed by a low pass filter of time constant \( 1/2AK \) and sensitivity.
\[ F_d = \frac{2\pi f_0 K_x}{A K} = \frac{2\pi}{f_0} \]  \hspace{1cm} (8.96)

Note that \( K_x \) is expressed in volts per Hertz.

For a second order PLL, the detected signal is observed at the output of the loop filter and the filter transfer function is taken of the form \((1 + sT_f)(1 + sT_1)\). The choice of this type of filter helps in minimizing the phase error and is made on the basis of the Wiener optimization technique. This technique was first applied by Lehav and Parks [7]. We will also discuss optimization techniques in Chapter 9. The output of the filter network is given by

\[ v_2 = \frac{\Delta q}{K_s} \sqrt{1 + (\cos(\omega_m t) - 2\Delta q/AK)} \cdot \cos(\omega_m t + \theta) \]  \hspace{1cm} (8.97)

where

\[ \theta = \arctan \frac{\sin(2\Delta q - \omega_m AK)}{\omega_m} \]

Thus the output of the filter gives the demodulated output. Before we comment on the capability of a PLL to act as an FM demodulator let us consider the magnitude of the phase error (cf. 8.92), given by

\[ |v_3| = \frac{\Delta q \sqrt{\omega_m^2 + (\omega_m^2 AK)^2}}{\sqrt{\omega_m^2 - \omega_0^2} + 4\Delta q^2 \omega_m^2} \]  \hspace{1cm} (8.99)

This shows that the phase error changes with the modulating frequency. Thus if the phase error increases beyond \(\pi/6\), distortions will appear. Thus one has to carefully design the system so that this does not happen over the required range of modulating frequency.

To clarify this, let us consider the special case for which \( \omega_m = \frac{\sqrt{2}}{2} \).

Therefore, (8.99) can be written as

\[ |v_3| = \frac{\Delta q \sqrt{\omega_m^2 + (\omega_m^2 AK)^2}}{\sqrt{\omega_m^2 - \omega_0^2} + 4\Delta q^2 \omega_m^2} \]  \hspace{1cm} (8.100)

The maximum value will appear at a value of \( \omega_m \) given by

\[ \omega_m = -\left( \frac{\sqrt{1+T^2}}{T} - T \right)^{1/2} \]  \hspace{1cm} (8.101)
where

\[ y = \left( \frac{\alpha_a}{\mathcal{K}} \right)^3 \]  

(8.102)

Usually the loop gain is large compared to the loop natural frequency. Therefore, the maximum phase excursions, \( \phi_{\text{max}} \), given by

\[ \phi_{\text{max}} = \frac{1}{\mathcal{K}} \frac{\Delta \phi}{\Delta \omega} \]  

(8.103)

Fixing this maximum value of \( \phi_0 \) to \( \pi/6 \) (say), one finds that the maximum frequency deviation of the input wave relative to the loop natural frequency is given by

\[ \frac{\Delta \omega}{\omega_{\text{nat}}} = \frac{\sqrt{2} \pi}{6} \]  

(8.104)

Further referring to the practical case, for which the damping factor is \( \sqrt{2}/2 \), and the loop gain is large compared to the loop natural frequency, one finds that (cf. 8.97) the output of the loop filter is given by

\[ v_o = \frac{\omega_{\text{nat}}}{K} \sqrt{\frac{1 + \frac{2\alpha_a}{\omega_{\text{nat}}}}{\omega_{\text{nat}} + \omega_a}} \]  

(8.105)

where

\[ \theta = \arctan \left( \frac{\omega_{\text{nat}}}{\omega_a} \right) + \arctan \left( \frac{\omega_{\text{nat}}}{\omega_a} \right) \]  

(8.106)

If \( \omega_a \) is large compared to \( \omega_{\text{nat}} \), one finds that

\[ v_o = \frac{2\omega_{\text{nat}}}{K} \Omega \cos \omega_t \]  

In such a case, the PLL acts as an FM discriminator of gain

\[ K_f = \frac{2\omega_{\text{nat}}}{K} \]  

This is exactly similar to that of a first order PLL.

Now referring to the expression (8.97) one finds that phase-locked demodulator performance (when the phase detector operates in the linear region of its characteristic) is similar to that of a frequency discriminator followed by a filter of transfer function

\[ F_f(\phi) = \frac{\omega_{\text{nat}}}{\Omega} \left( 1 + \frac{\alpha_a}{\omega_{\text{nat}}} + \frac{\alpha_a}{\omega_{\text{nat}}} \right) \]  

(8.107)
The filter transfer function can be realized with the help of the network of Fig. 8.6. Note that 

\[ \frac{\omega_c^2}{L^2}, \frac{2\zeta_0 R}{L} \quad \text{and} \quad \frac{Z_0 - \omega_c AK}{\omega_c} = xCR. \]

![Fig. 8.6. A passive filter network.](image-url)

### 8.5 Nonlinear Operation of Phase Locked Demodulators

In this section we will consider the operation of a first order phase locked demodulator (PLD) when the phase excursion exceeds \( \pi/2 \).

We will compare its performance with that of a standard limiter discriminator, the characteristic of which is shown in Fig. 8.7. The phase locked demodulator characteristic is shown in Fig. 8.5. Let us consider that an FM wave of the form \( f \cos (\omega_c t + m_0 \sin \omega_d t) \) is fed to both the demodulators.

Let us further assume that the centre frequency of the limiter discriminator (LD) and that of the VCO are tuned to the carrier frequency of the FM signal. When the maximum frequency deviation \( \Delta f_{\text{max}} \) is small, the outputs of both the LD and PLD will be free from distortions. This is shown in Fig. 8.7a. Now when the maximum frequency deviation is such that it goes beyond the bent portions of the LD and PLD characteristics, nonlinear distortions will appear. For the LD, the output will be as shown in Fig. 8.7b.
Fig. 5.7. A similar discriminator characteristics.

Fig. 8.8. A phase-locked demodulator characteristics.
The figure clearly explains the waveform. However, for the PLD it requires certain explanation.

When $\Delta$ is large, i.e., when it is greater than $AK$, the instantaneous frequency error $\Delta \cos \omega_d t$ becomes greater than $AK$ at certain parts of the modulating cycle. As a result, the loop remains out of lock till the instantaneous frequency error drops down to $AK$. During the unlocked period, the output of the phase detector will be given by (cf. 7.56)

$$v_p = \Delta \left( x - \frac{x^2 - 1}{x + \cos (2b_0 - 2\alpha)} \right)$$

(8.109)

where

$$\Delta = \frac{\Omega}{AK}$$

$$2\alpha = AK\sqrt{\frac{3}{t^2} - 1 - (t^2 + 1)}$$

and

$$b_0 = \text{arcsec} \sqrt{\frac{3}{t^2} - 1}.$$
\sin \varphi = \varphi - \frac{q^2}{2} - \frac{q^3}{3!} - ... \quad (8.114)\\
The expression for \sin \varphi can also be empirically (3) written as\\
\sin \varphi = 0.583q - 0.135q^3 - 2.0 < q < 2.0 \quad (8.116a)\\
which fits in with the actual expression over the range of \sqrt{\varphi} lying within \pm 3.0 radians. Now assume that the input signal is a two tone modulated one, i.e.,\\
\varphi(t) = \frac{\Delta_{1}}{\sin \omega t} + \frac{\Delta_{2}}{\sin \omega t} \quad (8.111)\\
Therefore, the differential equation of the first order RLC is\\
\frac{d^2\varphi}{dt^2} + AK \sin \varphi = \Delta_{1} \cos \omega t + \Delta_{2} \cos \omega t \quad (8.112)\\
For the sake of convenience, we put\\
\Omega(t) = \Delta_{1} \cos \omega t + \Delta_{2} \cos \omega t \quad (8.113)\\
Assume the solution of \varphi as\\
\varphi = \psi(t) + h_{1} + h_{2} + h_{3} + h_{4} \quad (8.114)\\
where \psi(t) is a small parameter less than unity. Now introduce this parameter \psi(t) before the nonlinear terms in the following way,\\
\sin \varphi = \psi(t) \frac{\Delta_{1}}{2} + \psi(t)^2 \frac{\Delta_{2}}{2} \quad (8.115)\\
Substitute \varphi from (8.114) and \sin \varphi from (8.115) in (8.112) and equate like powers of \psi(t) to yield:\\
\psi^4 \text{ terms: } \frac{d\psi}{dt} A K \varphi(t) = \Omega(t) \quad (3.116a)\\
\psi^4 \text{ terms: } \frac{d\psi}{dt} A K \varphi(t) = 0 \quad (3.116b)\\
\psi^4 \text{ terms: } \frac{d\psi}{dt} A K \varphi(t) = AK \varphi(t)^2 \quad (3.116c)\\
\psi^4 \text{ terms: } \frac{d\psi}{dt} A K \varphi(t) = AK \frac{\varphi(t)}{2} \varphi(t)^2 \quad (3.116d)\\
\psi^4 \text{ terms: } \frac{d\psi}{dt} A K \varphi(t) = \frac{AK}{2} \frac{\varphi(t)}{2} \varphi(t)^2 - AK \frac{\varphi(t)}{2} \varphi(t)^2 \quad (3.116e)
From the above relations, it is clear that \( \gamma_1, \gamma_2 \) etc. are zero and \( \gamma_3, \gamma_4 \) etc. are given by

\[
\gamma_m = \frac{\Delta_{\gamma_m}}{\sqrt{\nu_{\gamma_m} + (AK)^3}} + \frac{\Delta_{\gamma_m}}{\sqrt{\nu_{\gamma_m}^2 + (AK)^3}}
\]

where

\[
\nu_{\gamma_m} = \frac{3\gamma_m}{4} \left( \gamma_m^2 + 2\nu_{\gamma_m} \right) \cos(\nu_{\gamma_m} - 2\nu_m) + \frac{3\gamma_m}{4} \left( \gamma_m^2 + 2\nu_{\gamma_m} \right) \cos(\nu_{\gamma_m} - 2\nu_m) + \frac{3\gamma_m}{4} \left( \gamma_m^2 + 2\nu_{\gamma_m} \right) \cos(\nu_{\gamma_m} - 2\nu_m) + \frac{3\gamma_m}{4} \left( \gamma_m^2 + 2\nu_{\gamma_m} \right) \cos(\nu_{\gamma_m} - 2\nu_m)
\]

\[
\nu_{\gamma_m} = \frac{\Delta_{\gamma_m}}{AK}
\]

\[
\nu_{\gamma_m} = \frac{\Delta_{\gamma_m}}{AK}
\]

and

\[
\nu_{\gamma_m} = \frac{AK}{4} \left[ \frac{3\gamma_m}{4} \left( \gamma_m^2 + 2\nu_{\gamma_m} \right) \cos(\nu_{\gamma_m} - 2\nu_m) + \frac{3\gamma_m}{4} \left( \gamma_m^2 + 2\nu_{\gamma_m} \right) \cos(\nu_{\gamma_m} - 2\nu_m) + \frac{3\gamma_m}{4} \left( \gamma_m^2 + 2\nu_{\gamma_m} \right) \cos(\nu_{\gamma_m} - 2\nu_m) + \frac{3\gamma_m}{4} \left( \gamma_m^2 + 2\nu_{\gamma_m} \right) \cos(\nu_{\gamma_m} - 2\nu_m) \right]
\]

\[
\nu_{\gamma_m} = \frac{\Delta_{\gamma_m}}{AK}
\]
Neglecting $q_2$ term, one finds that

$$\varphi = q_1 + q_2$$

(8.119)

Thus one finds that the output of the phase detector, which is proportional to $\sin \varphi$, consists of the following distortion terms:

1) self-distortions—components at frequencies $\omega_1$ and $\omega_2$ appearing in the expressions for $q_1$;

2) harmonic distortions—components at $3\omega_1$ and $3\omega_2$;

3) intermodulation distortions—components at $3\omega_1 \pm \omega_2$ and $\omega_2 \pm 2\omega_1$.

Let us now consider the response of a PLL incorporating an integrating filter of transfer function $F(\omega) = (1 + \mu \omega^2)/\omega$, to a frequency ramp signal. That is, we consider a PLL, which is initially in lock with an open-loop frequency error. And at time $t = 0$, the frequency of the carrier is made to vary linearly with time with a slope $R$, referring to (8.8) one easily finds that the loop equation can be written as

$$\frac{d\varphi}{dt} + 2\xi \omega_1 \cos \varphi \frac{d\varphi}{dt} + \omega_1^2 \sin \varphi = \frac{\mu R}{\omega_1^2}$$

(8.120)

which can be re-written as

$$\frac{d^2\varphi}{dt^2} + 2\xi \omega_1 \cos \varphi \frac{d\varphi}{dt} + \omega_1^2 \sin \varphi = \frac{R}{\omega_1^2}$$

(1.121)

This equation is numerically solved via a digital computer for $\varphi$, and the variation of $\varphi$ with time is shown in Fig. 8.9 and compared with the case when sin $\varphi$ is replaced by $q_1$, i.e., for the linearized PLL. Note that Fig. 8.9a shows the case when $\omega_1$ is small so that the steady state phase error is less than $\pi/6$, whereas, Fig. 8.9b depicts the case for which the steady state phase error is more than $\pi/2$. From the Fig. 8.9 one finds that when the nonlinear section of the PLL is taken into account, the overshoots and phase settling times become longer compared to those of a linearized PLL. Moreover, the frequency of the resultant oscillation of a non-linear PLL is smaller than that of a linearized PLL.
Fig. 8.9. Transient responses of a second-order PLL to a frequency ramp signal having different values of $K_i$: (a) $K_i=0.5$, (b) $K_i=0.8$. 

- Transient response of a second-order PLL to a frequency ramp signal having different values of $K_i$. 
- (a) $K_i=0.5$, (b) $K_i=0.8$. 

The diagrams illustrate the transient response of a second-order PLL to a frequency ramp signal with varying $K_i$ values. The graphs show the phase response over time for different $K_i$ values, highlighting the system's behavior under these conditions.
6.6 Reception of a Doppler Shifted FM Signal

Now consider the reception of a signal that has undergone Doppler shift of the form shown in Eq. (8.2). To approximate this, let us consider that the Doppler shift appears in the form of a frequency shift. Therefore, the instantaneous frequency of the received signal, which is a frequency modulated one, is given by (8.231)

\[ \nu(t) = \frac{d}{dt} \Psi(t) \]

(6.127)

It is to be noted that although both pulse time expansion and the bandwidth compression will occur due to Doppler effect, yet it will be small compared to the carrier shift. And we do not include this into the equation here. Therefore, the received signal in the ground station is

\[ r(t) = A \sin (\omega_0 t - \int \nu(t) \, dt + \phi_0 \sin \omega_0 t) \]

(6.123a)

where

\[ \Psi(t) = m_0 \sin \omega_0 t - \int \nu(t) \, dt \]

(6.124)

Let us consider the case when the signal is radiated from an accelerating satellite in which case

\[ \nu(t) = \dot{R} t \]

(6.125)

Therefore

\[ \Psi(t) = m_0 \sin \omega_0 t - \frac{1}{2} \dot{R} t^2 \]

(6.126)

Thus the instantaneous frequency deviation of the received signal is

\[ \frac{d\nu(t)}{dt} = \Delta \omega_0 \sin \omega_0 t - \dot{R} t \]

(6.127)

Thus if one proposes to use a standard discriminator, then once he has to adjust the coarse frequency of the discriminator continuously in synchronization with the lightning motion of the satellite. Otherwise, the discriminator will become out of tune and the reception will fail through. Manual adjustment of tuning is impossible. Thus a standard LD cannot be used for the reception of such signals.
Let us consider a PDL with an integrating filter and assume that the coarse frequency of the VCO is equal to the carrier frequency. Therefore, if the phase error is not large, the output of the phase detector is given by

\[ v_p = AE_k E_p = \frac{PT(F)}{\sigma + AE_k} \]

(8.128)

For the type of the signal, as given in (8.127), one gets

\[ v_d(t) = \frac{\text{Sin}}{v^2 + \omega_0^2} \]

(8.129)

Substituting this in (8.128) and taking the filter network of (8.49), one finds that in the steady state, the output of the detector is given by

\[ v_p = \frac{v_0}{E_k} \sqrt{c_0^2 + c_0^2 m_0^2 + \frac{4E_k^2 c_0^2}{c_0^2 - c_0^2}} \cos \left( \omega_0 t + \frac{2\pi c_0^2}{c_0^2 - c_0^2} \right) - \frac{E_k}{\omega_0^2} \]

(8.130)

If \( \omega_0 \) is close to \( \omega_m \), one finds that

\[ v_p = \frac{v_0}{E_k} \frac{\Delta \omega_0}{2 E_k} \cos \omega_0 t - \frac{E_k}{\omega_0^2} \]

(8.131)

The first part of the right hand side of (8.131) gives the demodulated signal. Thus a PDL can be used to demodulate an FM signal from an accelerating satellite. If one takes the nonlinear loop, this offset voltage due to the Doppler shift, if large, will introduce distortion.

REFERENCES

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LINEAR ANALYSIS WITH STOCHASTIC INPUTS

In the preceding Chapter, the response of a linear PLL to a pure signal, modulated with deterministic signals was considered. In the real world, the picture is different. The signal, that is fed to the input of a PLL, is always accompanied by unwanted disturbances, and in many situations the modulating signal assumes the character of a random signal. In this chapter, we propose to take up the study on the response of a linear PLL to such signals.

Before we take up the case, let us estimate the extent of contamination of the signal by the unwanted disturbances. Take the case of a space communication system, where the transmitter and the receiver are separated for apart. Note that the extent of corruption not only depends on the strength of the unwanted disturbances, but also on the attenuation suffered by the signal in arriving at the receiver end.

To estimate the amount of attenuation, let us consider an artificial satellite located at a distance of \( r \) meters and transmitting at a wavelength of 2 meters with the help of an aerial having a directive gain \( G_r \). Therefore, the received power at the ground station is:

\[
P_r = \frac{P_t G_r G_a}{4\pi r^2} A_a
\]  \hspace{1cm} (9.1)

where \( P_r \) is the power radiated by the transmitting antenna. \( A_a \) is the antenna aperture of the receiving aerial and \( G_r \) is the directive gain of the receiving antenna. Therefore, the transmission loss is given by:

\[
L = L_r = \frac{4\pi r^2}{G_r} \frac{1}{A_a}
\]  \hspace{1cm} (9.2)
Now noting that

$$A_s = \frac{G_s^2}{4\pi}$$  (8.3)

one gets from (9.2)

$$L = \left(\frac{4\pi^2}{\lambda^2}\right) \cdot \frac{1}{G_s G_r}$$  (9.1)

Therefore, the transmission or path loss in dB is

$$L_{\text{transmission}} = 20 \log_{10}(\text{G}_s) + 20 \log_{10}\lambda - 10 \log_{10}G_r - 10 \log_{10}G_s$$  (9.2)

Note that $\lambda$ and $\lambda$ are in meters.

Expressing the distance $r$ in Km and frequency in GHz one gets

$$L_{\text{transmission}} = 92.5 + 20 \log_{10}(r) + 20 \log_{10}f$$

$$- 10 \log_{10}G_r - 10 \log_{10}G_s$$  (9.7)

For a typical active communication satellite the maximum value of $r$ is

$$r_{\text{max}} = \sqrt{a + R^2} = a$$  (9.7)

where $a$ is the radius of the earth and $R$ is the height of the satellite from the surface of the earth. Take, for example, the case of INTELSAT—IV F6 and F8. A typical value of $r_{\text{max}}$ is 41680 Km. and $f$ the lower limit of the satellite communication band, is about 3.76 GHz. The values of $G_s$ and $G_r$ are of the order of 1 and 10$^6$ respectively. Thus

$$L = 156 \text{ dB}$$  (9.8)

Thus the strength of the received signal is very low, and such any unwanted disturbances that may accompany the received signal will considerably affect the reception of the signal. The strength of the received signal is even lower for deep space communication. The sources of the unwanted disturbances are many (1). a.m. transmitter noise, antenna noise, galactic noise, man-made noise, cosmic input noise, etc. Thus the net signal at the input to the receiver may be written as

$$N(t) = \sqrt{2A\sin(2\pi f + \Phi)} + N_c(t)$$  (9.8)

$N(t)$ takes into account the effect of all sources of unwanted disturbances. For the sake of simplicity we will assume that $N_c(t)$ has the character of thermal noise. Moreover, in practice it has been found that this assumption also gives a fairly accurate estimate of
the system performance. Using quantum mechanics, Nyquist and others have shown that for a system, which is in equilibrium at temperature $T^\circ K$, the energy associated with each degree of freedom is [Chapter 1]

\[ E = \frac{1}{2} \exp(\frac{h'}{kT}) - 1 \tag{9.10} \]

where \( h' \) is the Planck's constant and is given by

\[ h = 6.625 \times 10^{-34} \text{ joule-seconds} \]

and \( k' \) is the Boltzmann's constant and is given by

\[ k = 1.38 \times 10^{-23} \text{ joule-degree Kelvin} \]

Since \( kT/h \) is $6.25 \times 10^{-8}$ Hz at 300 degree Kelvin, it can be shown from (9.10) that, for \( f \ll kT/h \), which is usually true

\[ E = kT/2 \tag{9.11} \]

This expression is valid up to 100 GHz. Thus we find that the spectral density of the noise that accompany the signal is flat and the spectral density is $kT/2$. This is the so-called two-sided power spectral density and the single-sided power spectral density is given by

\[ N_s = kT \text{ Watts/Hz} \tag{9.12} \]

Note that since the noise is frequency independent, it is called white noise. It is to be noted that an equivalent value of $T$ can be assigned by measuring the total noise power at the receiver end.

It is interesting to note that in a well-designed S-band communication system the equivalent noise temperatures due to different sources have the following typical values:

- Galactic and radio source noise: $4K$
- Atmospheric noise: $2K$
- Earth temperature noise: $1^\circ K$
- Antenna and transmission loss: $20^\circ K$
- Amplifier noise: $10^\circ K$

Thus the total equivalent noise temperature is about $37^\circ K$. However, there are other sources of noise and the above mentioned different noise temperatures also vary and as such the typical receiver noise temperature is taken of the order of $60^\circ K$. 
9.1 Characterization of Input Noise

Now at the receiver input, all noise outside the band of the signal is eliminated by passing the received signal $R(t)$ through a (square cut-off) bandpass filter having a bandwidth $B$. The centre frequency of the bandpass filter is tuned to the carrier frequency of the received signal. The output of the bandpass filter can be written [7] as

$$n(t) = \sqrt{2} n_1 (1 \cos \omega_0 t + \sqrt{2} n_2 (1 \sin \omega_0 t)$$  \hspace{1cm} (9.13)

The spectrum of $n(t)$ is illustrated in Fig. 9.1.

![Fig. 9.1. Spectral density of bandlimited noise.](image)

Some properties of the narrow-band noise expansion are summarized below for the sake of convenience (Chapter 1):

1. The spectra of $n_1(t)$ and $n_2(t)$ are low pass in nature.
2. If $n(t)$ is Gaussian, then $n_1(t)$ and $n_2(t)$ are also Gaussian.
3. If $n(t)$ is a zero mean process, then so also are $n_1(t)$ and $n_2(t)$.
4. $n_1(t)$ and $n_2(t)$ are independent, i.e.,

$$E[n_1(t)n_2(t)] = 0$$  \hspace{1cm} (9.14)

5. Variance of $n(t)$ is identical to that of either $n_1(t)$ or $n_2(t)$, i.e.,

$$\mathcal{E}(t) = n_1^2(t) = n_2^2(t)$$  \hspace{1cm} (9.15)

6. Spectral densities of $n_1(t)$, $n_2(t)$ and $n(t)$ are given by

$$S_{n_1}(\omega) = \frac{N}{2} \left\{ \begin{array}{ll} 1 & \text{if } -\frac{B}{2} < |\omega| < \left( \frac{B}{2} - \frac{B}{2} \right) \\ 0 & \text{otherwise} \end{array} \right.$$  \hspace{1cm} (9.16)

and

$$S_{n_2}(\omega) = \frac{N}{2} \left\{ \begin{array}{ll} 1 & \text{if } -\frac{B}{2} < |\omega| < \left( \frac{B}{2} - \frac{B}{2} \right) \\ 0 & \text{otherwise} \end{array} \right.$$  \hspace{1cm} (9.17)
\[ S_k(\nu) = S_0(\nu) = N_0B \text{ for } |f| < \frac{B}{2} \]  
\[ = 0 \text{ for } |f| > \frac{B}{2} \]  
Equations (9.18) and (9.19) indicate that \( n_o(t) \) and \( n_d(t) \) have low pass character of bandwidth \( B/2 \) and spectral density \( N_0B \).

9.2 Derivation of the Loop Equation in the Presence of Additive Noise

The input to the PLL, when the signal is corrupted with additive noise, is given by
\[ R(t) = \sqrt{2} A \sin (w_0t + \Psi_1(t)) + n(t) \]  
where \( n(t) \) is given by (9.13); \( n(t) \) may be further rewritten as
\[ n(t) = \sqrt{2} N_0(t) \cos (w_0t + \Psi_3(t)) - \sqrt{2} N_1(t) \sin (w_0t + \Psi_3(t)) \]  
where
\[ N_0(t) = n_o(t) \cos \Psi_1 + n_d(t) \sin \Psi_1 \]  
and
\[ N_1(t) = n_o(t) \cos \Psi_1 - n_d(t) \sin \Psi_1 \]

Now writing the output of the voltage controlled oscillator as
\[ R_d(t) = \sqrt{2} E_0 \cos (w_0t + \Phi(t)) \]  
one finds that the output of the phase detector is
\[ v_p = E_i E_0 [A \sin \varphi + N_j \cos \varphi - N_k \sin \varphi] \]  
where
\[ \varphi = (w_1 - w_0) t + \Psi_3 - \Psi_2 \]  
It is worthwhile to note that in deriving the above equation, the inherent oscillator noise has been neglected. Therefore, the output of the filter network is given by
\[ v_d(t) = E_i E_0 \left[ (A \sin \varphi(t) + N_j(t)) f(t - u) \right] \]  
where
\[ N_j(t) = N_j \cos \varphi - N_k \sin \varphi \]  
Hence
\[ \frac{dv}{dt} = KF(p) \{4 \sin \varphi + N(t) \} \]  
(9.29)

\[ \frac{dy}{dt} = (\omega_s - \omega_0) - KF(p) \{4 \sin \varphi + N(t)\} + \frac{dv}{dt} \]  
(9.30)

where \( p \) is the differential operator.

Putting
\[ z_1 = \Psi_1 + (\omega_s - \omega_0) t \]  
(9.31)

can rewrite (9.29) as
\[ \frac{dz_1}{dt} = -KF(p) \{4 \sin \varphi + N(t)\} + \frac{dv}{dt} \]  
(9.32)

Considering the equations (9.29) and (9.31), the mathematical model of the PLL is shown in Fig. 9.2. In order that the model could be utilized for calculation, it is necessary that the nature of \( N(t) \) be ascertained. To do this, let us rewrite \( N(t) \) as
\[ N(t) = \sqrt{2} \sigma_s \cos (\omega_s - \omega_0) t - \Psi_1 - \sqrt{2} \sigma_s \sin \]  
(9.33)

Referring to (9.33) one finds that the determination of the power spectral density of \( N(t) \) is difficult, since a part of \( \Psi_1 \) is derived from \( N(t) \). But it has been shown [3] that if \( \sigma(t) \) has a symmetrical band-pass spectral density which is much wider than the bandwidth of \( \Psi_1 \), then the loop noise \( N(t) \) has the same spectral shape as that of \( \sigma(t) \) except that it is a low pass version of \( \sigma(t) \). That is, \( N(t) \) is Gaussian noise with one-sided spectral density \( N_e \).
222 Phase Lock Theories and Applications

Normally the operation of the loop is carried out by tuning the VCO frequency to the carrier frequency, i.e., \( v = \omega_c \) therefore, (9.32) reduces to

\[
\frac{dv}{dt} = \frac{dv}{dt} - K_F(p)[A \sin \theta + N(p)] \tag{9.34}
\]

9.3 Mean Square Phase Error and Noise Bandwidth

We again assume that the loop is operating in such a way that the phase error excursion is less than \( \pi/6 \). And thus we can replace \( \sin \theta \) by \( \theta \). This means that the modulation index of the input angle modulated signal as well as the noise level is small. Hence we re-write (9.34) as

\[
\frac{dv}{dt} = \frac{dv}{dt} - K \int \frac{dH(x)}{dt} f(t - u) du \tag{9.36}
\]

Taking the Laplace transform of (9.34a) one gets

\[
\mathcal{V}(s) = \frac{s \mathcal{V}(0) + K_F(s) N(s)}{s + \alpha K_F(s)} \tag{9.35}
\]

It is to be noted that the first term on the right hand side of (9.35) indicates the modulation phase error, whereas the second term gives the noise phase error. Further note that the VCO phase modulation is obtained as (cf. 9.35)

\[
\mathcal{V}_c(s) = \mathcal{V}(s) - \mathcal{V}(0) - \frac{K_F(s) N(s)}{s + \alpha K_F(s)} \left( \frac{AK_F(s)}{s + AK_F(s)} \right) \mathcal{V}(s) + \frac{N(s)}{s} \tag{9.36}
\]

The first term on the right hand side of (9.36) gives the VCO phase modulation due to the signal component whereas the second term gives the phase modulation of the VCO due to the noise component.

Remembering that \( N(s) \) is Gaussian random noise with one-sided spectral density \( N_0 \), it is readily seen from (9.36) that the mean square phase modulation due to the noise, i.e., the variance of the VCO modulation due to noise, is given by
\[ a_{\text{tot}} = \frac{N_0}{2 \Delta f} \int_{-\Delta f}^{\Delta f} \frac{AKF(b)}{1 + AKF(b)} \, dB \]  
\[ (3.37) \]

The limits of integration are restricted between \(-B/2\) and \(+B/2\) because the input noise process is passed through a baseband filter of bandwidth \(B\) centred about \(b_0\).

Let us now take the filter networks of (8.42) and (8.43) and write

\[ H(b) = \frac{AKF(b)}{\Delta f + AKF(b)} \]  
\[ (3.38) \]

as

\[ H_1(b) = \frac{AK(1 + b T)}{T_1 \Delta f + AKF(1 + b T)} \]  
\[ (3.39) \]

and

\[ H_2(b) = \frac{AK(1 + b T)}{T_1 \Delta f + (1 + AK) T} \]  
\[ (3.40) \]

i.e.,

\[ H_1(b) = \frac{\alpha^2 + 2L \alpha b + b^2}{\alpha^2 + 2L \alpha b + b^2} \]  
\[ (3.41) \]

and

\[ H_2(b) = \frac{\alpha^2 + 2L \alpha b + b^2}{\alpha^2 + 2L \alpha b + b^2} \]  
\[ (3.42) \]

Therefore,

\[ \left\{H_1(b)\right\}^2 = \frac{\alpha^2 (\alpha + 4L \alpha^2 b^2)}{(\alpha^2 - \alpha 0^2) + 4L \alpha^2 b^2} \]  
\[ (3.43) \]

and hence \(N_p\) for the PLL with the proportional plus integrating filter is given by

\[ N_p = \frac{N_0}{2 \Delta f} \int_{-\Delta f}^{\Delta f} \frac{\alpha^2 (\alpha + 4L \alpha^2 b^2)}{(\alpha^2 - \alpha 0^2) + 4L \alpha^2 b^2} \, dB \]  
\[ (3.44) \]

where \( W = 3\alpha 0R \).

This is the variance of the noise phase modulation for a loop with an imperfect integrator, \(N_p\), for a loop with perfect integrator can be easily computed by putting \( \xi = 0 \) in (9.44) (cf. 9.41 and 9.42).

Equation (9.44) can be written as

\[ N_p = \frac{N_0}{2 \Delta f} \int_{-\Delta f}^{\Delta f} \frac{\alpha^2 (\alpha + 4L \alpha^2 b^2)}{\alpha^2 + 2L \alpha b + (2L \alpha^2 b^2 - \alpha 0^2) + \alpha^2} \, dB \]  
\[ (9.45) \]
The roots of the denominator polynomials
\[ x_1^2, x_2^2 = \left(1 - 2l_1^2 \right) \pm \sqrt{4 \left(l_1^2 - 1\right) l_1^2} \]  
(9.47)

Note that
\[ l_1 = l_2 - \frac{1}{2l_1^2 \sinh x} \]  
(9.48)

If \( l_1 > 1 \), then put
\[ l_0 = \cosh \frac{a}{2} \]
\[ l_0^2 = 1 - \sinh^2 \frac{a}{2} \]
\[ 1 - 2l_0^2 = 1 - 2 \cosh^2 \frac{a}{2} = - \cosh a \]
Thus
\[ x_1^2 = x_2^2 = - \cosh a \pm \sinh a \]
\[ = - e^{-a} - e^{a} \]

and hence
\[ \varphi_{2n} = \frac{N e^{\pi i/2}}{A 2^n} \int_0^1 \frac{1 + 4l_0^2 x^2}{(x^2 + e^{-a}) (x^2 + e^{a})} \, dx \]
\[ = \frac{N e^{\pi i/2}}{4n! (e^{a} \sinh \frac{a}{2})} \left[ 1 - 4l_0^2 \left( e^{-a} \right) \tan^{-1} \left( x \cosh a \right) \right] \]
\[ - \left( e^{-a} - 4l_1^2 \right) \tan^{-1} \left( x \cosh \frac{a}{2} \right) \]  
(9.49)

where \( x_0 = \frac{W}{2n} \).  
(9.50)

If we assume that \( W > 2n \), then
\[ \varphi_{2n} = \frac{N e^{\pi i/2}}{A 2^n} \left( 1 - 4l_1^2 e^{-a} + 4l_1^2 \right) \]
\[ = \frac{N e^{\pi i/2}}{8 \pi} \left( 1 + 4l_1^2 \right) \frac{1}{\cosh \frac{a}{2}} = \frac{N e^{\pi i/2}}{8 \pi} \left( 1 + 4l_1^2 \right) \]  
(9.51)

i.e.,
\[ \varphi_{2n} = \frac{N e^{\pi i/2}}{8 \pi} \left( 1 + 4l_1^2 \left( 1 - \frac{1}{2l_1^2 \sinh x} \right) \right) \]  
(9.52)
\[ n_0^* = \frac{2N_f}{\pi d^2} \left[ \frac{1}{2e_0} \left( 1 - \frac{1}{2e_0 a^2 f^2} \right) \right] \tag{9.53} \]

when 
\[ e_0 < 1, \quad \text{put} \quad e_0 = \sin (q/2) \]

Thus one writes
\[
\begin{align*}
& \frac{1 + 4e_0^2 a^2}{a^2 + 2e_0^2 (2e_0^2 - 1) + 1} - \frac{(1 - 4e_0^2) + 4e_0^2 (a^2 + 1)}{a^2 - 4e_0^2 \cos (q/2) + 1} \\
& = \frac{(1 - 4e_0^2)}{a^2 - 2x \cos (q/2) + 1} - \frac{2x - 2 \cos (q/2)}{a^2 - 2x \cos (q/2) + 1} \\
& + \frac{4e_0^2 (a^2 + 1) \cos (q/2)}{a^2 - 2x + 2 \cos (q/2) + 1} \\
& + \frac{(a^2 - 2x + 2 \cos (q/2))}{(a^2 - 2x + 2 \cos (q/2))} \cdot \frac{\cos q/2}{\cos q/2} \\
& = \frac{1 + 4e_0^2}{2x - 1} \frac{1 + 1}{4 \sin^2 (q/2)} \tag{9.54}
\end{align*}
\]

Therefore, from (9.46) and (9.54) one gets
\[
\begin{align*}
& n_0^* = \frac{N_f}{d^2} \int \left[ \frac{1 + 4e_0^2 a^2}{2x - 1} \arctan \frac{W^2 - 4e_0^2}{W \sin q/2} \right. \\
& \left. + \frac{1 - 4e_0^2}{W^2 - 4e_0^2 \cos (q/2) + 4e_0^2} \cdot \frac{\cos q/2}{\cos q/2} \right] dx \tag{9.55}
\end{align*}
\]

If one assumes that \( W \to \infty \), then (9.55) reduces to
\[ n_0^* = \frac{N_f}{d^2} \int \left[ 1 + 4e_0^2 \right] dx \tag{9.56} \]

which is similar to (9.53).

In practice, it is found that in most situations, the integrand of (9.44) assumes a negligible value beyond \( W/2 \). And as such (9.46) may be approximately written as
\[ n_0^* = \frac{N_f}{d^2} \int \left[ 1 + 4e_0^2 a^2 \right] dx \tag{9.57} \]
\[ x_1^2, x_2^2 = \cos \varphi + j \sin \varphi = \exp (\pm j\varphi) \]  

\[ z_1 = e^{j\varphi}, z_2 = -e^{j\varphi}, z_3 = e^{-j\varphi}, z_4 = -e^{-j\varphi} \]

Thus we find that the roots of the denominator polynomials are complex conjugate. This is shown in Fig. 9.3.

\[ \text{Lim}_{r \to 0} H(c) = 0 \]

where

\[ H(c) = \frac{1 + 4c + c^2}{c^2 + 2c (e^{\theta} - 1)} + 1 \]

From the theory of contour integration, one finds

\[ R_\infty [H(c), z_1] = \frac{1 + 4e^{\theta} + e^{2\theta}}{2c e^{\theta} (2j \sin \varphi)} \]

\[ R_\infty [U(c), z_1] = \frac{1 + 4e^{\theta} + e^{2\theta}}{2c e^{\theta} (2j \sin \varphi)} \]

From (9.58), (9.59) and (9.61) it is seen that...
\[ \Phi_0^* = \left( \frac{N_0}{\Delta f} \right) \frac{1}{2} \int \left[ 1 + \frac{4\epsilon_i^2}{\epsilon_1} \right] \frac{1 + \frac{4\epsilon_i^2}{\epsilon_1}}{2\epsilon_1^2} \sin^2 \phi \, d\phi \]

That is,

\[ \Phi_0^* = \frac{N_0}{\Delta f} \frac{1}{\sin \phi} \cos \left( \frac{\phi}{2} \right) \left( 1 + \frac{4\epsilon_i^2}{\epsilon_1} \right) \]

\[ \Phi_0^* = \frac{N_0}{\Delta f} \frac{1}{8 \sin \left( \frac{\phi}{2} \right)} \left[ 1 + 4\epsilon_i^2 \right] \quad (9.62) \]

Therefore, from (9.62) and (9.48) one gets

\[ \Phi_0^* = \frac{N_0}{\Delta f} \frac{1}{2} \left[ \frac{1}{2\epsilon_1^2} + \frac{1}{2\epsilon_1^2} \right]^T \]

\[ \Phi_0^* = \frac{N_0}{\Delta f} \frac{1}{4\epsilon_1^2} \left( 1 + 4\epsilon_i^2 \right) \quad (9.63) \]

This is exactly similar to the expression in (9.53).

The phase error variance for a PLL incorporating a perfect integrator can be found by putting \( \epsilon_i = \epsilon_1 \) in (9.51) or (9.63) and one obtains

\[ \Phi_0^* = \frac{N_0}{\Delta f} \frac{1}{4\epsilon_1^2} \left( 1 + 4\epsilon_1^2 \right) \quad (9.64) \]

Noise bandwidth is defined as (Chapter 1)

\[ B_n = \int \left| H(j\omega) \right|^2 \frac{1}{\Delta f} \, d\omega \quad (9.65) \]

\[ B_n = \int \left| H(j\omega) \right|^2 \frac{1}{\Delta f} \, d\omega \quad (9.65a) \]

\[ B_n = \int \left| H(j\omega) \right|^2 \frac{1}{\Delta f} \, d\omega \quad (9.65b) \]

As we have already stated that response of the loop-filter drops to zero beyond \( B_n \Delta f \), then (9.57) may be written as

\[ \Phi_0^* = \frac{N_0}{\Delta f} \int_{-\infty}^{\infty} \left| H(j\omega) \right|^2 \frac{1}{\Delta f} \, d\omega \quad (9.66) \]

Thus comparing (9.66) and (9.65) one finds

\[ \Phi_0^* = \frac{N_0}{\Delta f} B_n \quad (8.67) \]
Referring to (9.67) one finds that $B_a$ (single-sided) is the bandwidth of an ideal square cut-off low-pass filter that produces the same amount of noise power output as does the linearised phase locked loop. The concept of noise-bandwidth helps one to gather information regarding the propagation of noise through a phase locked loop. 

The amount of labour involved in computing the phase error variance and hence the noise bandwidth ($B_a$) with the help of the integrals (9.66) and (9.67) can be reduced considerably by using the following results [4].

$$I_a = \frac{1}{2\pi j} \int_{-\infty}^{\infty} f(s) f(-s) \, ds \quad (9.68)$$

where

$$f(s) = \frac{C_1 + C_2 s + \cdots + C_{2n} s^n}{s^{n+1} + \cdots + s s^n} \quad (9.69)$$

Then

$$i = \frac{C_1^2}{2\pi \rho \rho_0} \quad (9.68a)$$

$$I = \frac{C_1^2 \rho_0}{2\pi \rho} \quad (9.68b)$$

$$I_a = \frac{C_1^2 \rho_0}{2\pi \rho} + \frac{C_1^2}{2\pi \rho_0} - 2C_1 C_2 \frac{\rho_0}{2\pi \rho} + \frac{C_2^2 \rho_0}{2\pi \rho} \quad (9.68c)$$

The variation of the noise bandwidth of a PLL, incorporating an integrating filter, with $\xi$ is shown in Fig. 9.4. The expression for $B_a$ in this case easily is seen to be given by (cf. 9.64 and 9.67)

$$B_a = \frac{\sqrt{2}}{2} \left( \xi + \frac{1}{\xi} \right) \quad (9.70)$$

Obviously, the minimum value of $B_a$ appears when $\xi = 1$. This is obtained by putting $\frac{d}{d\xi} B_a = 0$. The minimum value of $B_a$ is given by

$$B_{a\text{min}} = \frac{\sqrt{2}}{2} \quad (9.71)$$

By referring to Fig. 9.4, one finds that when $\xi$ lies between 0.25 and 1, $B_a$ does not exceed its minimum value by more than 25 per cent. This gives the freedom of varying the values of the damping factor over a range around 0.5.
Another parameter that is commonly used is the 'signal-to-noise ratio in the loop bandwidth'. This is defined as

\[ (SNR)_E = \frac{A^2}{N_0(B_n)} - \frac{A^2}{N_0B_n}. \]  

(9.72)

\( B_n \) denotes the noise bandwidth. Comparing (9.72) and (9.67) one finds that

\[ (SNR)_E = \frac{1}{3B_n} \]  

(9.73)

9.4 Optimization of Loop Performance

From what we have learned until now, we can conclude the following:

1. Modulation tracking error becomes smaller for larger values of the open loop gain. This indicates that the open loop gain of the loop bandwidth should be made as large as possible in order to achieve the best tracking performance.

2. The propagation of noise through the loop, that causes phase jitter of the VCO, is smaller for lower values of the loop bandwidth.

3. So far as the linear operation of the loop is concerned, the behaviour of the loop can be studied by considering it as a linear filter with transfer function.
having \( \Psi(t) \) as its input (cf. (9.36). The properties (1) and (2) suggest that the best modulation tracking property and the best noise squelching performance of the loop cannot be realized simultaneously. Thus some sort of a compromise is needed. What is then the criterion of goodness? The criterion, Jaffe and Rechtin [5] chose, a minimization of the mean square loop error, defined as

\[
E^2 = X^2 \sigma_t^2 + \sigma_e^2
\]

where \( \sigma_t^2 \) is the total transient error, defined as

\[
\sigma_t^2 = \int \sigma_t^2(t) \, dt
\]

(9.15)

Here \( \sigma_e(t) \) is the instantaneous loop phase error due to transients. The quantity \( X^2 \) is the Lagrangian multiplier, a design parameter that is used to adjust the proportions of noise and transient phase error in the loop. Note that \( \sigma_e^2 \) denotes the variance of the noise phase error. Using the Werner optimization technique [6], Jaffe and Rechtin found that \( X^2 \) will be minimized provided the loop transfer function satisfies the following relation [Chapter 1].

\[
U_d(x) = 1 - \sqrt{\frac{\dot{\nu}^2}{\nu(t)^2}}
\]

(9.76)

This is incidentally, the Yovitis-Jackson [7] formula. Note that \( \dot{\nu}^2 \) appears because of the equivalent noise to \( H(\nu) \) is \( N(\nu)/\nu \). Now if the signal modulation \( \Psi(t) \) is of the type

\[
\Psi(t) = a_0 + a_1 t + a_2 t^2 + \ldots
\]

(9.77)

where \( a_0, a_1, a_2, \ldots \) are independent random variables, it can be shown that

\[
S(\nu) = \nu^2. E \left[ \frac{\ddot{D}_F(\nu)}{D_F(\nu)} \right] + \dot{\nu}^2 \nu^2 \theta^2
\]

(9.78)

where \( E(\cdot) \) is the statistical expectation operator, and \( D_F(\nu) \) is the Laplace transform of \( \Psi_F(\nu) \). The plus (+) superscript in (9.76) implies taking those terms in the partial fraction expansion of \( S(\nu) \) that have left-half plane poles and zeros.
9.4.1 Optimum Filter for a Phase Step Signal

In this case for a phase step of size $\Phi$, one gets

$$S(\phi) = \frac{N_t \Phi^2}{\pi^2} \frac{N_t}{2A^2}$$

That is,

$$S(\phi) = \frac{10 \Phi^2}{\pi^2} + \frac{N_t}{2A^2} \quad (9.79)$$

Hence,

$$H(\phi) = 1 - \frac{\sqrt{N_t} \phi^2}{\pi} + \frac{\sqrt{N_t} \phi \Delta}{\pi}$$

$$= \frac{2 \lambda \Delta}{\pi} \sqrt{N_t}$$

(9.81)

Thus referring to (9.81) one finds that $H(\phi)$ is the transfer function of a first order PLL with

$$\Delta K = 2 \lambda \Delta \sqrt{N_t}$$

(9.82)

$$\omega_n = 2 \lambda \Delta \sqrt{N_t}$$

(9.83)

9.4.2 Optimum Filter for Frequency Step Signal

In this case, one finds that

$$S(\Omega) = \frac{N_t \Omega^4}{\pi^4} + \frac{N_t}{2A^4}$$

(9.84)

where $\Omega$ is the size of the frequency step.

Now putting

$$B_t = 2(\Omega \Delta)^2 / N_t$$

(9.85)

one gets

$$S(\Omega) = \left( \frac{1}{B_t} + \frac{1}{B_{t+c}^2} \right) \frac{N_t \Omega^4}{\pi^4}$$

(9.86)

and

$$\Omega(\Omega)_t = \left( \frac{1}{B_t} + \frac{1}{B_{t+c}^2} \right) \frac{N_t \Omega^4}{\pi^4} \Delta$$

Therefore, the optimum loop transfer function is given by
Comparing this with the transfer function of a second order PLL with an integrating filter, i.e.,

\[ H_0(s) = \frac{s^2 + 2\zeta_0 \omega_0 s + \omega_0^2}{s^2 + \frac{1}{\lambda} \omega_0^2 s + \omega_0^2} \]

one finds that

\[ \omega_0^2 = \sqrt{3} \lambda \Delta \Omega \sqrt{N_0} \]

\[ \xi_0 = \sqrt{2/3} \]

Thus by writing the loop transfer function in terms of \( \omega_0 \) and \( \xi_0 \), one eliminates the Lagrange multiplier (\( \lambda \)).

9.4.3 **Optimum Filter for a Frequency Ramp Signal**

For a signal that is linearly frequency modulated, one gets

\[ \Psi_0(t) = \frac{R_0}{A} \]

i.e.,

\[ S_0(t) = \frac{3A}{\Delta f} \frac{N_0}{2A} \]

Putting

\[ R_0^2 = 2(\Delta f)^2 N_0 \]

i.e.,

\[ R_0^2 = \frac{\sqrt{2} \Delta f \Delta R N_0}{\Delta f} \]

one finds that

\[ S_0(t) = 3 A R_0 \left( \frac{1}{\Delta f} + \frac{1}{\Delta f} \right) \]

i.e.,

\[ S_0(t) = \frac{3 A R_0}{\Delta f} \left( \frac{1}{\Delta f} + \frac{1}{\Delta f} \right) \left( \frac{1}{\Delta f} + \frac{1}{\Delta f} \right) \times \left( \frac{1}{\Delta f} + \frac{1}{\Delta f} \right) \]

Consequently,

\[ (S_0(t)) = 3 A R_0 \left( \frac{1}{\Delta f} + \frac{1}{\Delta f} \right) \left( \frac{1}{\Delta f} + \frac{1}{\Delta f} \right) \]

\[ = 3 A R_0 \left( \frac{\Delta f^2 + 2\Delta f^2 + \Delta f^2}{\Delta f^2} \right) \]
Therefore,

\[
H(z) = 1 - \frac{\sqrt{N_0/2}A^2}{\beta K \left( \frac{p^2 + 2pB_0^* + 2B_0^* + B_0^*}{p^2 + 2B_0^* + B_0^*} \right)} \tag{9.90}
\]

\[
= \frac{2pB_0^* + 2B_0^* + B_0^*}{p^2 + 2B_0^* + B_0^*} + B_0^* \tag{9.90}
\]

Compare this with the transfer function of a PLL incorporating a filter network

\[
F(z) = \frac{(1 + zT_1)(1 + zT_1)}{p^2T_1^2} \tag{9.91}
\]

i.e.,

\[
H(z) = \frac{\delta AKT_1}{2T_1^2} + \frac{\delta AK(T_1 + T_2)}{T_1^2} + \frac{\delta AK}{T_1^2} \tag{9.92}
\]

One finds that for the case of a frequency ramp signal, the optimum loop is a third order PLL with a filter network of the form of (9.91) where

\[
\begin{align*}
B_0^* &= \frac{AK}{T_1T_2} \\
2B_0^* &= \frac{AK(T_1 + T_2)}{T_1T_2} \\
2B_0^* &= \frac{AKT_1}{T_1T_2}
\end{align*}
\]

Therefore, the filter transfer function for the optimum loop is

\[
F(z) = \frac{1 + zT_1^2 + zT_1 + T_1}{p^2T_1^2} \tag{9.93}
\]

i.e.,

\[
F(z) = \frac{B_0^* + 2B_0^* + 2B_0^*}{AK} \tag{9.93}
\]

9.4.4 Optimum Tracking for a Frequency Step Signal Accompanied with Random Phase Offset

Here we consider the optimum filter design for a frequency step signal with a uniformly distributed phase offset \(\psi_0\) between \((-\pi, \pi)\). This case was considered by Tansworth [8]. In such a situation one finds that
\[
D_\varepsilon(s) = \frac{\gamma_s}{s + \frac{\beta}{\Delta}}
\]

Noting that \(E(Y_a) = \mu_3^a, \), one gets
\[
E(D_\varepsilon(s) D_\varepsilon(-s) = \frac{\beta}{\Delta} - \frac{s^2}{\Delta 3}
\]

Hence
\[
S(\sigma) = K \left( \frac{\mu_1^a}{\sigma} - \frac{\mu_2^a}{\sigma^2} \right) + N_0 2\Delta
\]

Posing
\[
\sigma^2 = \sqrt{\frac{\Delta}{4\Delta(\eta)}} \sqrt{\frac{\Delta}{\eta}}
\]

one can rewrite \(S(\sigma)\) as
\[
S(\sigma) = \sigma \sqrt{\frac{\Delta}{4\Delta(\eta)}} \left( \frac{\mu_1^a}{\sigma^2} - \frac{\mu_2^a}{\sigma^4} + \frac{\beta}{\Delta} \right)
\]

Thus
\[
(\sigma \eta) = \frac{\mu_1^a}{\beta_1} \left( \frac{\mu_1^a}{\beta_1^2} + \frac{\mu_1^a}{\beta_1^3} + 2\beta_1^2 \right)^{\frac{1}{2}} \sigma + \frac{\beta_1}{\beta_1^2}
\]

Hence, the optimum loop transfer function is given by
\[
H_d(s) = \left( \frac{\mu_1^a}{\beta_1^2} + 2\beta_1^2 \right) s + \frac{\beta_1}{\beta_1^2} \quad (9.94)
\]

Compare this with the transfer function \(H_d(s)\) of a PLL with an integrating filter, as given by
\[
H_d(s) = \frac{\epsilon_0^2 + 2\epsilon_0 \omega \epsilon_0}{\epsilon_0^2 + 2\epsilon_0 \omega \epsilon_0 + \omega_0^2}
\]

to eliminate the Lagrange multiplier from the design parameter in terms of \(\epsilon_0\) and \(\omega_0\). Here one finds
\[
\epsilon_0 = \sqrt{\beta_1(\mu_1^a)} \sqrt{\frac{\Delta}{\eta}} = \beta_1^2
\]
\[
2\epsilon_0 \omega_0 = \left( \frac{\epsilon_0^2}{\omega_0^2} + 2 \right) \epsilon_0 \quad (9.95)
\]

From the foregoing analysis we find that:

1) The optimization procedure is valid for a linearized loop with additive gaussian noise.
2) The type of the filter network to be incorporated depends on the nature of the input modulation.

3) The values of the filter parameters, defined in terms of the loop damping \( k \) and the loop natural frequency \( \omega_n \), depend on \( \Delta fN_0 \), i.e., on the input carrier-to-noise ratio. This means that an optimum loop gives optimum tracking so long the input carrier-to-noise ratio remains fixed. However, this is not true in practice. A method of realizing the dependence of the filter parameters on the input carrier-to-noise ratio is either to proceed a PLL with a bandpass limiter or to incorporate an automatic gain control circuit in the loop.

9.5 Loop Parameters Selection in Practice for Frequency and Phase Offset

In practice the frequency offset, \( \Delta f \), is not usually known before synchronization. But it has been known, the loop could have been easily locked by tuning the VCO. As such, it is reasonable to assign a random character to \( \Delta f \). Moreover, the received phase \( \phi_r \) is also a random variable uniformly distributed in the interval \( (-\pi, \pi) \).

The previous calculation can be easily applied to this case, provided the symbol \( \phi \) is taken as \( \sqrt{E(f)} \). We have already seen that in this case optimum filter network has the following structure

\[
F(s) = \frac{1 + T_0 s}{T_1 s}
\]

which requires an operational amplifier for its realization. The optimum loop transfer function then looks like

\[
H(s) = \frac{\omega_0^2 + 2z \omega_0 s \Delta f}{\omega_0^2 + 2z \omega_0 s \Delta f + \omega_n^2}
\]

Since it is easier to design a passive filter network, one would try with a passive filter, having the transfer function

\[
F(s) = \frac{1 + T_0 s}{1 + T_1 s}
\]

and end up with the following expression for the loop transfer function...
This will approach the optimum loop transfer function, provided
\[ AKT_0 \gg 1 \] (9.96)
Even with this value of \( AKT_0 \), the loop will introduce a steady state phase error,
\[ \theta_0 = \frac{\Theta}{K} \] (9.97)
which does not appear in the case of the real optimum loop using an integrating filter. However, this phase error, \( \theta_0 \), can be eliminated by the VCO after it has achieved locking.
Further, it is desirable that the loop noise bandwidth should be made small in order to minimize propagation of noise through the loop. Usually, the loop noise bandwidth is made much smaller than \( \Omega \). Obviously, in such a case a good lock-in behaviour is not expected. Thus in order to achieve phase locking in this circumstance, the frequency of the VCO is slowly swept by controlling the input voltage to the VCO through an external agency. That the loop will acquire the signal in a small amount of time as \( B_0 \) as \( \Omega \), the r.m.s. frequency offset, becomes less than \( 2\pi B_0 \). Therefore, inserting the condition
\[ \Omega = 2\pi B_0 \] (9.98)
in the expression for \( B_0 \), when \( AKT_0 \gg 1 \), one gets
\[ \Omega = 2\pi B_0 = \pi \Omega_0 \left( \frac{\xi_0}{4\Omega_0^2} \right) \] (9.99)
Again referring to (9.95) one finds that
\[ 2\xi_0 = \left( \frac{\pi \Omega_0^2}{4\Omega_0^2} + 2 \right)^{1/2} \] (9.100)
Eliminating \( \frac{\xi_0}{2\Omega_0^2} \) from (9.99) and (9.100), one finds that
\[ \xi_0 = 0.76 \] (9.101)
The corresponding optimum loop transfer function is
\[ H(s) = \frac{1 + (0.821/\Omega_0)s}{1 + (0.821/\Omega_0)s + (0.28/\Omega_0^2)s^2} \] (9.102)
This compass favourably with that, designed by Jaffe and Recehin,
\[ H_a = \frac{1}{1 + (0.75/R_a) + (0.285/R_a)^3} \]
referring to (9.79) and using Parseval's theorem, the expression for the transient error can be written as
\[ e(t) = \frac{1}{2\pi} \int_0^T \left[ 1 - H_a(j\omega) \right]^2 |D_v(j\omega)|^2 \, d\omega \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - H(a))(-H(-\beta)) |D_v(a)D_v(-\beta)| d\alpha \]
(9.104)

Note that
\[ E[D_v(a)D_v(-a)] = \frac{\sigma^2}{4\pi} - \frac{\sigma^2}{3\pi} \]
and for an integrating filter
\[ F(a) = \frac{1 + T_e a}{2\pi} \]
\[ 1 - H(a) = \frac{\sigma^2}{\sigma^2 + 2\pi T_e a + \sigma^2} \]
Therefore, using the above results, (9.104) and (9.68) one finds
\[ e(t) = \frac{\sigma^2}{4\pi} \left[ 1 + \frac{\sigma^2}{3\pi} \right] \]
(9.105)

For an optimum loop, using (9.99) and (9.101), it is seen that
\[ \theta(t) = \frac{2.085}{K_a} \]
(9.106)

If one takes the example of Jaffe and Recehin, it can be shown that
\[ \theta(t) = \frac{2.7}{K_a} \]
(9.107)

9.6 PLL Demodulator and Standard Limiter Discriminator

In this section, we will first consider the demodulation of a two-tone sinusoidal FM signal and then take up the study of the demodu-
latter of an FDM-FM signal. As usual, before the signal is fed to the demodulators, the signal is passed through a square cut-off bandpass filter, the bandwidth of which satisfies Carson’s bandwidth relation for keeping the distortion within tolerable limit. We write the test signal input as (cf. 9.22)

\[ R(t) = \sqrt{2} d \sin (\omega_f t + \Psi_f(t)) + \sqrt{2} n_q(t) \cos \omega_f t - \sqrt{2} n_q \sin \omega_f t \] (9.108)

9.6.1 Test Tone Modulated Signal Applied to a Limiter

Discriminator (LD) (cf. Fig. 9.5a)

![Diagram of LD and FLD configurations](Image)

Fig. 9.5. Limiter discriminator (LD) and phase locked demodulator (PLD) configurations.

Refer to Fig 9.5a and write

\[ \Psi_f = m \sin \omega_f t \] (9.109)

Thus, we find that

\[ R(t) = \sqrt{2} (d + n_i \sin \Psi_f - n_q \cos \Psi_f) \sin (\omega_f t + \Psi_f) + \sqrt{2} n_q \cos \Psi_f + m \sin \Psi_f \cos (\omega_f t + \Psi_f) \] (9.110)

For a high value of the input carrier-to-noise, defined as,

\[ (CNR)_0 = \frac{\mathcal{A}}{N_0} \] (9.111)

one finds that \( R_d(t) \), the output of the bandpass limiter, is approximated as
$$R(t) = \sqrt{2} A \sin \left( w t + \varphi_1 + \frac{N(t)}{A} \right)$$  \hspace{1cm} (9.112)$$

where

$$N(t) = (n_1 \cos \varphi_1 + n_2 \sin \varphi_1)$$  \hspace{1cm} (9.113)$$

Note that $N(t)$ is Gaussian and has the one-sided spectral density $S_n$. In deriving the relation (9.112), arc tan $(\cdot)$ has been replaced by $\varphi$.

Therefore, the input to the filter, after the discriminator, having a low pass filter of the form,

$$F(x) = \frac{1}{1 + x^2}$$  \hspace{1cm} (9.114)$$

is given by

$$v_1(t) = K_o \frac{d}{dt} \left[ m_1 \sin w t + \frac{N(t)}{A} \right]$$

i.e.,

$$v_1(t) = K_o \left[ m_1 \cos w t + \frac{d}{dt} \left( \frac{N(t)}{A} \right) \right]$$  \hspace{1cm} (9.115)$$

Therefore, if the bandwidth of the filter is such that the signal passes without attenuation, then the signal power output is given by

$$\overline{v_1^2}(t) = \frac{1}{2} A^2 (\Delta w)^2$$  \hspace{1cm} (9.116)$$

and the noise power output is

$$\overline{v_n^2}(t) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \overline{v_n^2}(t) dt$$  \hspace{1cm} (9.117)$$

where $v_n(t)$ is the noise component at the output of the filter using Fano's theorem, one obtains

$$\overline{v_n^2}(t) = \frac{1}{2 \pi} \int_{-\infty}^{\infty} |\nu(t)|^2 \, ds$$  \hspace{1cm} (9.118)$$

Since the input bandwidth extends between $(\Delta w, 2\Delta w)$, one finds that (9.118) reduces to

$$\overline{v_n^2}(t) = \frac{1}{2 \pi} \int_{-\Delta w}^{\Delta w} |\nu(t)|^2 \, ds$$  \hspace{1cm} (9.119)$$
Writing this in terms of the noise power input to the filter, one finds

\[
\frac{v^2}{\sigma^2}(t) = K^2_n \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{N_0}{2B} \frac{\omega^2}{1 + \omega^2 F_0^2} \, d\omega
\]  

(9.120)

In deriving the above result, use is made of (9.114), (9.115), and (9.119). Supposing that bandwidth of the filter is limited up to \( \omega_m \), one finds from

\[
\frac{v^2}{\sigma^2}(t) = K^2_n N_0 2\pi \int_{-\infty}^{\infty} \frac{\omega^2}{1 + \omega^2 F_0^2} \, d\omega
\]

\[
= K^2_n N_0 2\pi \left[ \arctan(\omega_m F_0) - \omega_m F_0 \right]
\]  

(9.121)

Normally, \( \omega_m T \ll 1 \), thus (9.121) reduces to

\[
\frac{v^2}{\sigma^2}(t) = K^2_n N_0 \omega^2 \omega_m^2
\]

(9.122)

(using the identity \( \arctan(x) = \omega_m T = \omega_m T - \frac{1}{3} \omega_m^3 T^3 \)).

Therefore, signal-to-noise power output is given by

\[
(SNR)_{DF} = K^2_n A^2 \Delta \omega \omega^2 \omega_m^2
\]

(9.123)

According to Carson's rule of thumb, one gets

\[
2\pi B = 2 \Delta \omega \omega_m
\]

(9.124)

Again for a wideband FM, one notes that \( \Delta \omega \gg \omega_m \). This (9.124) reduces to

\[
\omega_B = \Delta \omega
\]

(9.125)

Hence (9.123) can be written as

\[
(SNR)_{DF} = 3 \pi B^2 (CNR)
\]

(9.126)

6.2.3 TEST-TIME MODULATED SIGNAL FEED TO A PHASE LOCKED LOOP

Let us consider Fig. 9.5(g). The output of the PLL is passed through a low pass filter of the form (9.144). To begin with, we
consider a first order PLL. Assuming the linear operation of the PLL, the net output can be written as

$$v_o(t) = \frac{4\phi(t) + N(t)}{1 + st}$$  \hspace{1cm} (9.127)

where $\phi(t) + N(t)$ is the output of the phase detector and

$$\phi(t) = \frac{2\phi'(t)}{s + AK} - \frac{KN(t)}{s + AK}$$

Therefore, the output is given by

$$v_o(t) = \frac{4\phi'(t) - K\phi(t) - N(t)(1 + \Delta K)}{(s + AK)(1 + st)}$$  \hspace{1cm} (9.128)

If $\Delta K \gg \psi_{max}$ one finds that

$$v_o(t) = \frac{4\phi'(t)}{s + AK}$$  \hspace{1cm} (9.129)

Therefore, the signal power output is

$$\frac{\overline{v_o^2}}{\overline{v_s^2}} = \frac{1}{2AK}\Delta\alpha_b$$  \hspace{1cm} (9.130)

and the noise power output is

$$\frac{\overline{v_n^2}}{\overline{v_s^2}} = \frac{N_0}{2AK}\frac{1}{s^2 + \omega_b^2}$$  \hspace{1cm} (9.131)

Again assuming that the output filter has bandwidth of $\omega_{max}$ one finds that

$$\frac{\overline{v_o^2}}{\overline{v_s^2}} = \frac{N_0}{2AK}\int_{-\omega_b}^{\omega_b} \frac{1}{1 + \omega^2\omega_b^2} d\omega$$  \hspace{1cm} (9.132)

i.e.,

$$\frac{\overline{v_o^2}}{\overline{v_s^2}} = \frac{N_0}{2AK}\left[\pi/2\right]$$

Therefore, the signal-to-noise power ratio at the output is

$$\text{SNR}_{SP} = \frac{3\pi\Delta f^2}{N_0B} \frac{\Delta f}{\Delta B}$$  \hspace{1cm} (9.123)
242 Phase Lock Theories and Applications

t.e. \[ (\text{SNR})_{\text{rms}} = 3\sigma_{y}^{2} (\text{CNR}) \] (9.134)

This is exactly similar to that of a limited discriminator.

Let us now consider a second order PLL, with the loop filter

\[ F(s) = \frac{1 + T_1}{1 + T_2} \] (9.135)

The output of the filter \( F(s) \) is fed to the output filter. Therefore, one writes

\[ v(t) = \frac{1}{T_1} \left( \frac{A_T}{T_1} s + N(t) \right) \left( \frac{1}{T_2} s^2 + \frac{1}{T} (1 + AKT) s + \omega_n^2 \right) \] (9.136)

where

\[ 2\omega_{\text{OA}} = \frac{AKT}{2} \quad \frac{AK}{T} = \omega_{\text{in}}^2 \text{ and } AKT \gg 1 \]

Rewrite (9.135) as

\[ v(t) = \frac{1}{T_1} \frac{s (A_T s + N(t))}{s^2 + 2\omega_{\text{OA}} s + \omega_n^2} \] (9.137)

Take \( \xi = 1/\sqrt{2} \), and write the signal power output as

\[ \frac{v^2(t)}{T_1} = \frac{A_T^2}{2T_1} \frac{(\Delta f)^2}{\omega_{\text{OA}}^2 + \omega_n^2} \] (9.138)

The noise power output is given by

\[ \frac{v^2(t)}{T_1} = N_0 \frac{1}{2\pi T_1} \int_{-f_0}^{f_0} \frac{\omega^2}{\omega^2 + \omega_n^2} \, df_0 \]

\[ -N_0 \frac{1}{2\pi T_1} \int_{-f_0}^{f_0} \frac{\omega^2}{\omega^2 + \omega_n^2} \, df_0 \] (9.139)

If \( \omega_{\text{OA}} \gg \omega_{\text{in}} \), one finds that

\[ \frac{v^2(t)}{T_1} = \frac{N_0 \omega_{\text{OA}}}{2\pi T_1} \omega_n^2 \] (9.140)

Therefore, the signal-to-noise power ratio at the output is
\[ (\text{SNR})_{\text{PLL}} = 3 \left( \frac{\Delta f}{v_{\text{ref}}} \right)^2 \left( \frac{A_0}{\sigma_0} \right) \left( \frac{\text{SNR}}{1+\text{SNR}} \right) \]  

This is also similar to that of the first order loop. Thus at high CNR, the performance of a limiter discriminator is similar to that of a phase locked demodulator.

In designing a phase locked demodulator, the loop parameters should be chosen in such a way that the total loop phase error is kept to a minimum value so as to optimize the performance of the loop. The procedure for doing this is illustrated in the next section.

9.7 Optimum Phase Locked Demodulation of an FM Signal

In this section we will develop the structure of an optimum phase locked demodulator, when the input to the loop is a frequency modulated signal corrupted with additive Gaussian noise. Moreover, we will assume that the modulating signal is also taken to be a Gaussian variable with zero mean and rational spectral density [9].

These types of modulating signals are characteristic of many physical processes.

We have already seen that for faithful operation of the phase locked loop, the phase error should be small. Thus a linear version of a PLL will not be a bad approximation provided the noise level is not high. Again, we have also seen that the output of the loop filter will not be an exact replica of the modulating signal. In order to achieve this, it becomes necessary to pass the loop filter output through an output filter, \( F_o(s) \), as shown in Fig. 9.6.

![Diagram](https://via.placeholder.com/150)

**Fig. 9.6.** The optimum phase demodulator model.
Let us denote the output of \( y_2(t) \) by \( y(t) \) and the input to the PLL by \( x(t) \). We have already seen that the input to the loop can be equivalently represented as

\[
x(t) = \Psi_1(t) + \frac{N(t)}{A}
\]  

(9.142)

Note that here \( \Psi_1(t) \) is a Gaussian process with zero mean. If the input signal is a frequency modulated one,

\[
\Psi_1(t) = m(t) \quad \text{(9.143)}
\]

Here \( m(t) \) is the modulating signal and \( a_f \) is the sensitivity of the frequency modulator.

Now in order to design an optimum phase locked demodulator, one must see that:

1. The loop must be first optimized on the basis of minimum mean square phase error, which leads to the optimum loop transfer function (see Chapter 1)

\[
H_e(t) = 1 - \frac{N(t)^2}{\mathcal{J}(t)}
\]

and the optimum loop filter is thus given by

\[
F(x) = \frac{aH_e(x)}{AK(1 - H_e(x))}
\]

(9.144)

ii) After having minimized the loop error variance, one has to minimize the variance of the demodulation error between the output of \( x(t) \) and \( m(t) \), i.e., \( E[\{m - \hat{m}\}^2] \).

In order to manipulate this conveniently we write

\[
H_e(x) = \frac{y(x)}{\Psi_1(x)}
\]

(9.145)

Further referring to (9.143) one finds that

\[
m(x) = \frac{2}{a_f} \Psi_1(x)
\]

(9.146)

Thus, the variance of the demodulation error is given by

\[
e^2 = E[\{m - \hat{m}\}^2] = \frac{1}{2a_f} \int \left( \frac{1}{a_f} - H_e(x) \right)^2 S_{X}(x) \, dx
\]
\[ H_a(s) = \frac{1}{S^*(s) L_s S_a(s) \Delta_s} \] (9.148)

where

\[ S_a(s) = S_T(s) + \frac{N_a}{2N} = S^*(s) S_a(s) \] (9.149)

\( S^*(s) \) contains all poles and zeros in the left half of the complex s-plane, and \( S_a(s) \) incorporates all the poles and zeros in the right half of the complex s-plane. \( G(s) \) denotes the transfer function of the plant having poles and zeros in the left of the complex s-plane.

To illustrate this, let us consider that the spectral density of the modulating signal is

\[ S_a(s) = \frac{2\theta}{\omega^2 + \omega^2} \] (9.150)

\[ S_T(s) = \frac{2\omega^2}{\omega^2 + \omega^2} \] (9.151)

Therefore,

\[ S(s) = S_T(s) + \frac{N_a}{2N} \] (9.152)

where

\[ b^* = 4\omega^2 N_s \] (9.153)
Putting \( b = \frac{\alpha}{\Delta} \) and \( f = \sqrt{1 + \frac{\alpha^2}{\Delta^2}} \), one finds that

\[
H_\Delta(x) = \alpha - \frac{\left( f - 1 \right) + \alpha f}{\Delta (\alpha + f + x^2)}
\]  
(9.154)

Therefore, the loop filter is given by (cf. 9.144 and 9.154)

\[
F(x) = \frac{\alpha}{\Delta} \frac{f(f-1) + \alpha f}{s + \alpha}
\]  
(9.155)

Now noting that

\[
S(x) = \frac{2\alpha s_f}{\alpha(s^2 + f^2) + \Delta x^2}
\]  
(9.156)

Thus

\[
S_x^2(x) = \frac{1}{2s^2} \frac{\alpha (\alpha + f) + s^2 x^2}{s^2 + f^2 + \alpha x^2}
\]  
(9.157)

and

\[
S_x^2(x) = \frac{\alpha (\alpha + f) - s^2 x^2}{s^2 + f^2 + \alpha x^2}
\]  
(9.158)

Therefore, referring to (9.148), (9.157), (9.18) one finds that

\[
\frac{\alpha s_x^2(s)}{\alpha x^2 + \Delta x^2} = \frac{1}{2s^2} \frac{\alpha (\alpha + f) + s^2 x^2}{s^2 + f^2 + \alpha x^2}
\]

i.e.

\[
H_\Delta(x) = 4s_x \frac{\alpha^2}{\alpha (\alpha + f) + s^2 x^2}
\]

\[
= \frac{\alpha}{\Delta} \left( f(f-1) + \alpha f \right)
\]  
(9.159)

Referring to Fig. 9.5 it is seen that

\[
H_\Delta(x) = \frac{x}{\Delta} \frac{\Delta}{x}
\]

i.e.

\[
F_x(x) = \frac{x}{\Delta} \frac{\Delta}{x}
\]

Hence from (9.148), (9.159) and (9.160) one finds that

\[
F_x(x) = \frac{\alpha}{\Delta (\alpha + f + x^2)}
\]  
(9.161)
The filter transfer functions $F_0(t)$ and $F_t(t)$ can be written as

$$F_0(t) = G_0 \frac{1 + \frac{t}{T_0}}{1 + \frac{t}{T_1}}$$

(9.163)

and

$$F_t(t) = G_t \frac{1 + \frac{t}{T_2}}{1 + \frac{t}{T_3}}$$

(9.163)

where

$$T_0 = \frac{T}{N_d}$$

$$T_1 = \frac{1}{a}$$

$$G = \frac{a}{AK}$$

and

$$G_t = \frac{\left(\frac{f_t}{f_0} - 1\right)^{b - K}}{2k \sigma_f}$$

(9.164)

Referring to the relation (9.164) one finds that the design of filter networks for the optimum operation of the loop requires a knowledge of the input carrier-to-noise ratio. Thus, depending on the carrier-to-noise ratio, the pole zero location will change. Thus a loop which is optimum for one value of the input carrier-to-noise ratio, is not optimum for other values of the input carrier-to-noise ratio. Thus to maintain optimum demodulation, when the input carrier-to-noise ratio varies, the filter network would have to be adaptive, i.e., the pole-zero location should change with input CNR.

The signal to noise ratio at the output of the optimum demodulator is defined as

$$S_N = \frac{1}{2N} \sum_{m=-N}^{N} S_m(s) d s$$

(9.165)

For the modulating signal with spectral density $2a(a^2 + a^2)$ one finds that

$$S_N = 1/2A$$

(9.166)

where $A$ denotes the value of $\tilde{a}$ for the optimum demodulator.

Since we have been designing a phase locked demodulator, one must be careful to check the value of mean square phase error, which is given by
$$\eta^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ 1 - H_b(\omega) \right] S_e(\omega) d\omega$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| H_b(\omega) \right|^2 \frac{N_0}{2} d\omega$$  \hspace{1cm} (9.167)

Referring to (3.154) and noting that (cf. (9.155))

$$S_e(\omega) = \frac{2a_0^2}{\omega^2 (\omega^2 + a_0^2)}$$

one finds that

$$\eta^2 = \frac{2a_0^2}{(f_f - f_i) \left( f_r - f_i \right)}$$  \hspace{1cm} (9.168)

Now recalling the relation (9.147) one finds that the optimum demodulation error variance is given by

$$\eta^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{T}{S_T} - H_{df}(\omega) \right|^2 S_e(\omega) d\omega$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| H_{df}(\omega) \right|^2 \frac{N_0}{2} d\omega$$

i.e.,

$$\eta^2 = \frac{3f - f_i + f_r + 1}{f_f[f_f - (f_r - f_i) + 1]} + \frac{4f^2 - 4f + 1}{f_f[f_f - 1]}$$

$$+ \frac{3f^2 - f_i^2 + f_r^2 + 1}{(f_f - 1)(f_f + 1)^2} + \frac{4f^2 - 4f + 1}{(f_f - 1)(f_f + 1)^2}$$  \hspace{1cm} (9.169)

Note that \( \eta^2 \) depends on the input carrier-to-noise ratio. The variation of \( \eta^2 \) with input carrier-to-noise ratio is shown in Fig. 9.7. Note that curves are drawn up to the value of \( A/\sqrt{N_0} \alpha \), at which \( \eta^2 = 1/4 \). This is because of the fact that if \( \eta^2 \) becomes greater than \( 1/4 \), the rate of slipping cycles (Chapter 10) becomes a significant factor and the linear analysis fails.

Further, it is important to note that the above results are valid for a loop incorporating adaptive filters \( F(z) \) and \( F_s(z) \). However, if fixed parameter filters are used, then the loop is designed at a suitable value of the carrier-to-noise ratio. The performance of the receiver at a value of the GNR, which is different from the design point, can be studied by adopting the method [10] as outlined above.
Fig. 9.7: Dependence of the phase error variance on the carrier-to-noise ratio.

Rewriting (9.12a) as

$$H_0(f) = \frac{a_0^2 + 2\gamma_0 f + o^2}{f^2 + 2\gamma_0 f + o^2}$$

where,

$$\gamma_0 = \frac{\sigma_n^2}{d^2}$$

$$2\gamma_0 f = \alpha f - 1)$$

$$2\gamma_0 o = \alpha d$$

$$f = 1 + \frac{3f}{d} \text{ (CNR)}^{1/2}$$

$$o^2 = 2\gamma_0 f \text{ (CNR)}^{1/2}$$

and

$$\text{CNR} = \frac{d^2(\sigma_n^2)}{d^2}$$

one finds that for large values of CNR, the loop damping factor becomes 1/\(\sqrt{2}\). That is, for the optimum loop, operating at high CNR, the transient response is also optimum.
Because of various physical processes and limitations of the circuit elements of the oscillator, the output is not spectrally pure and is represented as

\[ \sqrt{2} \Delta f (1 + k(t)) \cos (\omega_0 t + \phi(t) + \Delta \phi). \]

Here \( k(t) \) represents amplitude fluctuation; \( \Delta \phi \), being the normalized aging coefficient, gives long term instability.

It has already been pointed out in the beginning of this book that an oscillator is a regenerative feedback device comprising of a limiter type nonlinear element and a tuned amplifier. Because of the various physical processes, occurring in the active and passive elements of the amplifier and the limiter, noise corrupts the signal output. For example, if we break the loop at the point marked \( X \) (Fig. 9.8a)

![Diagram](image)

Fig. 9.8. The noisy oscillator model.
and introduce a signal of the form $\sin \omega_f t$ at this point, the output will be of the form $A \sin (\omega_f t + \Phi(t))$. Here $\Phi(t)$ is the phase jitter introduced by the amplifier-limiter chain. The spectral density of $\Phi(t)$ has been found to be of the form (12.18)

$$S_{\Phi}(\omega) = \frac{a_1}{\omega^2} + a_2 \quad (9.179)$$

The magnitude of the flicker noise constant, in the frequency range from 5 MHz to 100 MHz, has been found to be (12)

$$a_2 = F_0 \times 10^{-11} \text{ [rad}^2\text{]} \quad (9.171)$$

The noise factor $F_0$ depends on the transistor r.f. feedback and can be reduced to the order of $10^{-9}$.

The white noise constant $a_1$ is given by

$$a_1 = \frac{K_T}{\gamma_p R_0} \quad (9.172)$$

where $P_s$ is the signal power and $F_0$ is the noise factor.

Now assuming that the phase jitter is small, we can rewrite the output of the limiter-amplifier approximately as

$$A \sin \omega_f t + AN \cos \omega_f t$$

Note that $A \sin \omega_f t$ represents the input to the limiter. If we close the loop at the breaking point, thus forming an oscillator, we may assume that the output of the oscillator is of the form $A \sin (\omega_f t + \theta(t))$ and external signal $AN \cos \omega_f t$ is added to the loop as shown in Fig. 9.3b. Then the oscillator equation can be written as

$$Y(p) X(p) = A \sin (\omega_f t + \theta) - AN \cos \omega_f t$$

i.e.,

$$M(p) = \frac{1}{Y(p)} [A \sin (\omega_f t + \theta) - AN \cos \omega_f t] \quad (9.173)$$

where

$$\frac{1}{Y(p)} = 1 + \left( \frac{\omega_p}{\omega_f} + \frac{Z}{\omega_f^2} \right) \quad (9.176)$$

and $\gamma$ denotes a differential operator. Using the method of harmonic balances and assuming $\Phi$ to be small it is easily shown that

$$\frac{d\theta}{dt} = 2\omega \Phi + \frac{d\phi}{dt} \quad (9.175)$$

Therefore, the spectral density of $\theta$ is given by...
Phase Lock Theories and Applications

\[ S(\omega) = \left[ 1 + \left( \frac{\omega_0}{2Q\omega} \right)^2 \right] S_N(\omega) \]

This result agrees with the earlier works [12, 16, 17].

Using the results of \( S_N(\omega) \), it is easily shown [11, 12] that

\[ S(\omega) = \frac{H_1}{\omega^2} + \frac{H_2}{\omega} + \frac{H_3}{\omega} + H_4 \]

(9.176)

where

\[ H_1 = \frac{2\eta_0}{\omega^2} \]
\[ H_2 = \frac{\alpha_0^2}{\omega} \]
\[ H_3 = \frac{2\eta_0}{\omega} \]
\[ H_4 = \eta_0 \]

(9.177)

The values of \( \alpha_0 \) and \( \eta_0 \) are approximately given by

\[ \alpha_0 \approx 10^{-11} \]

and

\[ \eta_0 \approx 10^{-18} \]

\[ H_1 \Delta \text{Flicker frequency noise} \]
\[ H_2 \Delta \text{White frequency noise} \]
\[ H_3 \Delta \text{Flicker phase noise} \]
\[ H_4 \Delta \text{White phase noise} \]

For power limitation reasons, there should be upper and lower limitations of the frequency spectrum of the noise process \( \xi(t) \). In this case, the phase locked loop can be modelled as shown in Fig. 9.9. In this particular model, the centre frequency of the VCO is assumed to be in tune with the incoming signal. Now assuming a linearised version of the PLL, one can easily write

\[ \Psi_4(t) = \frac{4K_p E(t)}{p} (\Psi_3 - \Psi_4) \]
Therefore,
\[ \Psi(t) = \frac{AKF_0(t)}{s + AKF_0(s)} \Psi_0 - \frac{AKF_0(s)}{s + AKF_0(s)} \Theta(s) \] \hspace{1cm} (9.178)

The first part of the right hand side represents signal modulation of the VCO phase, whereas the second component of the right hand side of (9.178) represents fictitious VCO noise phase modulation.

The actual VCO phase noise modulation is given by
\[ \Theta(t) = \delta(t) - \frac{AKF_0(t)}{s + AKF_0(s)} \theta(s), \text{ if } \Psi_0 = 0 \]
\[ = (1 - H(s)) \theta(s) \] \hspace{1cm} (9.179)

Hence the variance of the VCO phase is given by
\[ \sigma^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |1 - H(s)|^2 |S_x(s)|^2 ds \] \hspace{1cm} (9.180)

If an integrating filter of the form
\[ F(s) = \frac{1}{s^2 } \]

is taken, then
\[ \sigma^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{s^2}{s^2 + 2\alpha^2 s + \omega_0^2} |S_x(s)|^2 ds \] \hspace{1cm} (9.181)
Although all the components in (9.175) add instability to the oscillator, yet the two most significant terms correspond to the flicker frequency noise and white frequency noise. This in the following:

the spectral density of the VCO phase noise will be modeled as

$$S(\omega) = \frac{H_k}{\omega^2} + \frac{H_k}{\omega^4}$$

(9.182)

$$S(\omega) = \frac{1}{2\pi} \int_{0}^{\infty} \frac{e^{-\frac{s}{\omega}}}{(s^2 + 2\omega_0^2)e^{-\frac{s}{\omega}}} \frac{dH_k}{ds} ds$$

$$+ \frac{1}{\pi} \int_{0}^{\infty} \left( \frac{\omega_0H_k}{\omega^2} \right) d\omega$$

(9.183)

The integration of the first integral is straightforward. The result is $H_k\omega_0^2$.
Integration of the second integral can be done in the following way:

For $\xi_0 > 1$

$$I = \int_{0}^{\infty} \left( \frac{\omega_0}{(\omega^2 + \omega_0^2)} \right) d\omega$$

where

$$\omega_0^2 = \omega_0^2 \xi_0 = \omega_0 \sqrt{\frac{(\xi_0)^2 - 1}{}$$

$$I = \frac{1}{4\omega_0^2} \sqrt{\xi_0^2 - 1} \int_{\xi_0}^{\infty} \frac{1}{\xi_0^2 + 1} d\xi$$

(9.184)

For $\xi_0 = 1$

$$I = \frac{1}{2\omega_0^2}$$

(9.185)

For $\xi_0 < 1$

$$I = \int_{0}^{\infty} \left( \frac{2\omega_0^2}{(\omega^2 + \omega_0^2)} \right) d\omega$$

$$= \frac{1}{4\omega_0^2} \sqrt{1 - \xi_0^2} \left[ \frac{1}{2} \arctan \frac{2\omega_0}{\sqrt{1 - \xi_0^2}} \right]$$

(9.186)
Therefore, from (9.183) to (9.186) one obtains
\[ h_R = H_s \frac{V_c}{4V_n} + \frac{H_s}{4V_n} \left( \frac{1}{2} \ln \frac{1 + \frac{2V_n}{V_c}}{1 - \frac{2V_n}{V_c}} - 1 \right) \]
for \( E_s = 1 \)
\[ = H_s \frac{V_c}{4V_n} + \frac{H_s}{2V_n} \]
for \( E_s = 1 \)
\[ = H_s \frac{V_c}{4V_n} + \frac{H_s}{4V_n} \left( 2 \ln \frac{1 + \frac{2V_n}{V_c}}{1 - \frac{2V_n}{V_c}} - 2 \right) \]
\[ = -2 \ln \frac{1 + \frac{2V_n}{V_c}}{1 - \frac{2V_n}{V_c}} \]
when the incoming signal is accompanied by additive noise of one-sided spectral density \( N_0 \), the total mean square phase error of the VCO output is given by
\[ \delta^2 = \frac{\sin^2 \left( \frac{1}{2} \right) N_0}{2 \pi^2} + \delta_s^2 \]
(9.184)
The coefficients of the different noise components \( h_n, H_s \) and \( H_n \)
which are functions of \( E_s \), are plotted against \( E_s \) in Fig. 9.10. From the plot one finds that the second term is a monotonic function of \( E_s \), which continuously decreases with \( E_s \). However, the first term of (9.188) attains a minimum value for \( E_s = \frac{1}{2} \). But the minimum does not exceed its minimum value by twenty-five percent, even if \( E_s \) is made to vary between 0.25 and 1.0 (cf. Section 9.3). Again, since the terms of (9.187) monotonically decrease with \( E_s \), it is reasonable to choose \( E_s = \frac{1}{\sqrt{2}} \), that gives the optimum transient response of the PLL. Thus putting \( E_s = \frac{1}{\sqrt{2}} \) in (9.188) one finds that
\[ \delta^2 = 3V_n \frac{V_c}{2V_n} + H_s \frac{V_c}{2V_n} - H_s \frac{V_c}{4V_n} \]
(9.186)
After having obtained this value of \( \delta^2 \) for \( E_s = \frac{1}{\sqrt{2}} \), one could minimize \( \delta^2 \) with respect to \( V_n \). Thus differentiating (9.188) with respect to \( V_n \) and putting \( \delta = 0 \), one finds the following equation for \( V_n \) that gives the least value,
\[ V_n^2 - \left( 2V_n^2 \frac{\delta h_s}{3N_0} \right) - \left( 2V_n^2 \frac{\delta h_s}{3N_0} \right) = 0 \]
(9.192)
Normally, for a highly stable crystal controlled oscillator \( V_n \) is
of the order of $10^{-6}$, and for typical design value of $A^2/N_0$ (say, of the order of $10^6$), it is seen that

$$27 \left( \frac{2\sqrt{2}AM}{3N_0} \right)^3 > 4 \left( \frac{2AM}{3N_0} \right)^3$$

i.e.,

$$H_2^2 > \frac{4}{81} H_1^2 \left( \frac{A^2}{N_0} \right)$$  \hspace{1cm} (9.191)

In this case, the value of $\omega_0$ is given by [12]

$$\omega_0 = \frac{2\sqrt{2}}{3} (H_1)^{1/2} \left( \frac{A^2}{N_0} \right)^{1/3} \cosh (\eta/3)$$  \hspace{1cm} (9.192)
\[
\cosh \phi = \frac{4.5\beta}{H_2} \sqrt{\gamma/(\beta/\gamma)}
\]  
(5.193)

\[
\beta = \frac{2\sqrt{\gamma}}{3} \left( \frac{H_2}{\gamma} \right)^{1/3} \left( \frac{\beta}{\gamma} \right)^{2/3} \cosh \frac{\gamma}{3}
\]  
(5.194)

where \( \cosh \phi = 4.5 \frac{H_2}{H_1} (\beta/\gamma)^{1/3} \)  
(5.195)

REFERENCES


CHAPTER 10
NONLINEAR BEHAVIOUR OF NOISE FREE LOOPS

In the preceding chapters we have tacitly assumed that the phase-locked loop is initially in lock and the phase error is small. The behaviour of the first order loop, whether it is in lock or out of lock, has been studied in detail in chapter 7. We now turn to the discussion of the nonlinear behaviour of higher order loops, which are most commonly used. On the other hand, a first order loop is rarely used in practice. However, a knowledge of the property of a first order loop does help in understanding the behaviour of higher order loops.

When a signal arrives at the input of a PLL, neither the frequency offset is precisely known, nor the phase difference is small. In such a situation it is necessary to examine the conditions under which a higher order can be made to lock on to the external signal. In the case of a first order loop, we have seen that if the open loop frequency error, i.e., the frequency offset, is less than the open loop gain of the loop (AK), the loop will achieve lock after a degree of time, called the locking time. We have further seen that the locking time depends on the open loop frequency error as well as on the open loop gain AK. This is not quite so for a higher order loop. To appreciate this, consider the operations of two second order loops, (which are commonly used) one incorporating a proportional plus integrating filter and the other an integrating filter. We know that the loop equations in these cases are given by

\[ \frac{d\phi}{dt} = 0 \]

(cf. 9.32 with \( A(\phi) = \theta \) and \( \frac{d\theta}{dt} = 0 \)).
208 Phase Lock Theorems and Applications

\[
T_1 \frac{d^2 \varphi}{dt^2} + (1 - AKT_1 \cos \varphi) \frac{d \varphi}{dt} + AK \sin \varphi = \Omega \tag{10.1}
\]

and

\[
T_2 \frac{d^2 \varphi}{dt^2} + AKT_2 \cos \varphi \frac{d \varphi}{dt} + AK \sin \varphi = 0 \tag{10.2}
\]

Note that (10.1) refers to a PLL with a proportional plus integrating filter, whereas (10.2) describes the operation of a PLL with an integrating filter, where the incoming wave is a pure CW signal without any modulation. Looking at (10.2) you easily conclude that locking will occur for any value of the open loop frequency error, because the steady state phase error does not depend on \( \Omega \).

Looking into (10.1), one finds that if the loop with the proportional plus integrating filter is to attain a steady state operation, then it is necessary that

\[ \varphi = \arcsin (Q AK) \]

This also indicates that the loop will never attain a steady state, i.e., it will not lock to the incoming signal, if the open loop frequency error (\( Q \)) is greater than the open loop gain (\( AK \)). Moreover, from (10.1) it cannot be often told that lock-in will always occur if \( Q \) is less than \( AK \). What is then the maximum value of \( Q \) up to which the loop can lock? How much time does it take to achieve lock? These are some of the questions, which we will consider in this chapter.

10.1 Signal Acquisition by a Second Order, Type-1 and Type-2 Loops

In the case of type-1 PLL, the loop equation is given by (10.1). We have already indicated that whether the lock-in will occur or not cannot be predicted by looking at the equation. This can only be done after achieving a solution of (10.1). Since it is a second order differential equation with periodic nonlinearity, there is no exact method of solution. As such the graphical technique via the phase plane portrait is employed. The method of plotting phase plane trajectories consists in drawing graphs in which the \( \varphi \)-axis represents instantaneous beat angular frequency and the \( \varphi \)-axis represents instantaneous phase difference. The phase plane plot of a first order loop is shown in Fig. 7.13, the plotting of which is straightforward because the instantaneous beat frequency \( \frac{d\varphi}{dt} \) is a simple function.
of the phase difference \( \psi \). Now to draw the phase plane trajectories of the second order PLL with the proportional plus integrating filter, we rewrite (10.1) in the following form:

Note that

\[
\frac{d\psi}{dt} = \frac{d}{d\psi} (\psi) = \frac{d}{d\psi} (\psi) \cdot \psi
\]

i.e.,

\[
\frac{d\psi}{dt} = \frac{d\psi}{d\psi} \cdot \psi
\]

This (10.1) can be rewritten as

\[
\frac{d\psi}{dt} = \frac{\Omega - AK \sin \psi - 1 + AKT_s \cos \psi}{T_1} \cdot \frac{\psi}{T_1}
\]

Putting

\[
e^2 = \frac{AK}{T_1}
\]

and

\[
e^2 \cdot T_s = \frac{AKT_s}{T_1}
\]

one finds that (10.3) reduces to

\[
\frac{d\psi_{(\text{new})}}{dt} = \frac{1}{\psi_{(\text{new})}} \left( \frac{\Omega}{AK} - \sin \psi \right) - 2 (e_s - \xi) \cdot \frac{\psi}{T_1} \cdot \cos \psi
\]

That is, it can be written as

\[
\frac{d\psi}{dt} = f(\psi)
\]

Note that \( \psi \) and \( i \) are functions of time.

Principles of drawing phase plane trajectories are illustrated in textbooks on control systems [1, 2]. However, for the sake of convenience, we state the following procedure for phase plane analysis:

a) The vertical axis of the phase plane corresponds to \( \dot{\psi} \) and its horizontal axis is calibrated in terms of \( \psi \), as shown in Fig. 10.1.
b) The trajectory on the phase plane is initiated at time \( t = 0 \). This initial point \( A \) is identified by noting

\[
\phi(0) = \phi|_{\omega=0} \quad \text{and} \quad \dot{\phi}(0) = \dot{\phi}|_{\omega=0}
\]

(10.6)

The position of the point \( A \) at a latter instant of time \( t = 0 + \Delta t \), i.e., the direction of the trajectory, is found by first noting the slope \( \frac{d\phi}{d\omega} \) at \( \phi(0), \dot{\phi}(0) \) with the help of the relation

\[
\frac{d\phi}{d\omega} = f_{\phi} \quad \text{(10.7)}
\]

where \( f_{\phi} = f(\phi(0), \dot{\phi}(0)) \)

and then calculating

\[
\Delta \dot{\phi} = f_{\phi} \cdot \Delta \phi
\]

(10.8)

i.e.,

\[
\dot{\phi}(t) = \ddot{\phi}(0) + f_{\phi} \cdot \Delta \phi
\]

\[
\phi(t) = \phi(0) + \Delta \phi
\]

(10.9)
Nonlinear Behaviour of Noise From Lamps, 363

For reason of accuracy, $\Delta\varphi$ should be taken as small as possible. Repeating this procedure one can find the position of the trajectory at a larger time $t_1$, $t_2$, etc., and thus complete the trajectory.

d) Note that the time interval between two points on the trajectory, say $A$ and $B$, can be calculated by using

$$\Delta t = \Delta \varphi$$

i.e.,

$$\Delta t = \Delta \varphi$$

(10.16)

Again, $\Delta\varphi$ should be taken as small as possible for reason of accuracy. Obviously, the total time elapsed in traversing a particular path may be calculated by adding such $\Delta t$'s.

By following this procedure, by adopting numerical calculation with the help of (10.4), phase plane trajectories are drawn which are shown in Fig. 10.2 and Fig. 10.3. In these figures, $\varphi$ has been limited to the boundary ($\pm \pi$, $\pi$) by folding all the trajectories onto this region. Further referring to these figures, one finds that when $\dot{\varphi}$ is positive, $\varphi$ increases and when $\dot{\varphi}$ is negative, $\varphi$ decreases. Hence, in the upper half plane, phase motion takes place from left to right, whereas phase motion is from right to left in the lower half plane. Refer to Fig. 10.2, and consider any initial condition, such as shown by the point $A$. As time increases, the point moves along the trajectory $AB$ until it comes to $C$. Then to record further movement one jumps back to $A$ and follows the phase motion starting with a value of $\varphi$ equal to that at $B$, and follows the trajectory $CD$ up to point $D$ and repeats back to $C$ and repeats the procedure.

In this way one finally comes to the point $L$, the lock-in point, representing the locked condition. By following the procedure as outlined, one finds that the locked condition is always achieved whatever be the initial conditions [3, 4], viz., for any values of $\varphi_0$ and $\varphi$. Whereas if one considers Fig. 10.3, one easily finds that roughly one does not come up to a lock-in point, unless the initial conditions are located in the lock-in slot. Figure 10.3 also shows that unless the loop satisfies the particular initial conditions corresponding to the lock-in slot, there is a limit cycle towards which all upper trajectories converge. Thus for the case of Fig. 10.3, the loop does not achieve lock-in condition.
If the ratio of the open loop frequency to the open loop gain, \( \Omega(\alpha) \), is such that the phase plane trajectories are of the form of Fig. 10.2, then the trajectories end at \( \psi = 0 \), whence:

\[
\psi = \arcsin \left( \frac{\Omega}{\Delta} \right) \quad \text{(10.11)}
\]

or

\[
\psi = 2\pi + \arcsin \left( \frac{\Omega}{\Delta} \right) \quad \text{(10.12)}
\]
Fig. 10.5 - The phase plane plots with $F(U) = U + F_k T_1 \phi(U + T_2 \phi$).

This indicates that the PLL may look around $\phi = \sin(\Omega \Delta t)$ on $2\pi + \sin(\Omega \Delta t)$.

Thus acquisition may be achieved with or without skipping cycles. Referring to Fig. 10.2, one finds that if the initial condition of the VCO is such that the starting point lies above the separation $AA$, the loop will skip cycles before settling down to the point given...
by (10.12). Whereas if the initial operating point lies below the separatrix, the loop will settle down to the point as given by (10.11) without slipping any cycle.

Thus by drawing a number of such phase plots for different values of $\Omega/\Delta \xi$, one can find out the limiting value of $\Omega/\Delta \xi$ for which the loop settles down to the steady state values of $\phi(0)$ and $\phi(0)$. This limiting value of $Q$, i.e., the open loop frequency error, is called the pulling range of the PLL. Obviously, this limiting value of $Q/\Delta \xi$ will be one for which the trajectories will decay, i.e., the $Q$ would be less than $Q(0) - Q(0)$, for any values of $Q(0)$ and $Q(0)$. Viterbi, exploiting the phase plane trajectories, arrived at the following expression for the pull-in range

$$\frac{Q}{\Delta \xi} = \sqrt{\frac{2}{T_p} \left( F_p + 2/\Delta \xi \right)^\frac{3}{2}}$$  \hspace{1cm} (10.13)$$

where $F_p = T_p / T_z$, is the ratio of the op to dc gain of the filter. In deriving the above expression (10.13) for the pull-in range, Viterbi made certain assumptions, which require that $Q$ changes very slowly within $-\pi$ and $+\pi$ of $Q$. This requires a large value of the time constant of the filter network.* We will explain this in the section when Richman's method [6] of deriving the expressions for the pull-in range and pull-in time will be illustrated.

Calculation of the acquisition time is rather difficult. However, Viterbi has shown [3] that when $T_z$ is large, the time required before no more cycles are skipped is approximately given by

$$T_f = T_z \left( \frac{\Omega}{\Delta \xi} \right)^\frac{3}{2}$$  \hspace{1cm} (10.14)$$

For a second order type two loop, i.e., a loop incorporating a perfect integrating filter, it is shown by referring to (10.2) that the loop equation is independent of the open loop frequency error. This means that the loop will attain locked state for any value of the open loop frequency error. This means that this type of loop has infinite lock range. This can also be seen by referring to phase plane trajectories of Fig. 10.4, which is obtained from (10.2) as

$$\frac{d(\phi)}{df} = \frac{\sin \phi}{\phi} - 2F_p \cos \phi$$  \hspace{1cm} (10.15)$$

*Obviously for low values of the filter time constant this result fails.
From (10.15) one finds that

\[ \frac{1}{4} \int_{-\infty}^{\infty} \phi \, dp = \int_{-\infty}^{\infty} \sin \, dp - \frac{2}{\pi} \int_{0}^{\infty} \phi \cos \, dp \]  

(10.16)
Using (10.15), one finds that the right hand side of (10.16) may be integrated by parts to yield

\[ \frac{1}{2\pi} \left[ \phi(\pi) - \phi(-\pi) \right] = -\pi \int_0^\pi \frac{1 - \cos 2\theta}{\theta} \, d\theta. \]  

(10.17)

Referring to the right hand side of (10.17), one finds that the result of integration will be positive for both the positive and negative values of \( \phi \). This means that during each cycle of \( \pi \) from \(-\pi\) to \(\pi\), \( \phi \) decreases for any initial value of \( \phi \), indicating that the pull-in range of a second order type-2 loop is infinite. Using the phase plane portrait, Vinter [3] has shown that the acquisition time of such a loop is approximately given by

\[ T_p = \frac{1}{2\pi} \left( \frac{\Omega}{\Delta k} \right)^2 = T_p \left( \frac{\Omega}{\Delta k} \right)^2. \]

(10.18)

It is to be noted that expressions for the acquisition time as given by \( T_p \) is not really the total acquisition time. This is the time taken by the loop to reach the separatrix \( \Delta k \), beyond which there is no further cycle slipping. This time is really the frequency pull-in time. Since there is no further cycle slipping beyond the separatrix \( \Delta k \), the time taken after \( T_p \) to reach near the steady-state values of \( \phi \) and \( \phi \) is said to be the phase pulling time, \( T_p \). Therefore, the acquisition time \( T_{acq} \) is given by

\[ T_{acq} = T_p + T_p. \]

(10.19)

### 10.2 Approximate Acquisition Analysis of a Second Order, Type-2, Loop

In this section, approximate expressions for the locking time and pull-in range will be derived. We have already seen that the locking time of a PLL can be thought of as consisting of the frequency pulling time and phase pulling time. Phase pulling time is the time taken by the loop to attain the lock conditions of phase between the input and the output if the initial frequency error is small. Frequency pulling time, on the other hand, is the time required for equalization of the frequency of the input and output brought about by a gradual increase of the steady phase detector output voltage.

During the period of phase pulling, the phase locked loop may be considered as a semi-direct current loop, while during the period of
frequency pulling the loop is a combined dc and ac one and hence gain characteristics assume a significant role. In the presence of frequency pulling there will be a steady drift of the differential frequency per beat cycle. The amount of drift is determined by the initial frequency error, and the gain of the open loop at the beat frequency. Thus the situation is best analysed in a sequence of beat cycles.

Further in this section we will assume that the PLL incorporates a proportional plus integrating filter, for which we have,

$$ P(\delta) = \frac{1 + T_d s}{1 + T_i s} = F_o + \frac{1 - F_o}{1 + T_i s} $$ (10.20)

where

$$ F_o = \frac{T_o}{T_i} $$ (10.21)

Using (10.20), we rewrite the loop equation (9.30) in the presence of noise and input modulation as

$$ \frac{d\delta}{dt} = (\Omega - A K F_s \sin \phi) - A K F_s \sin \phi $$ (10.22)

where,

$$ F_s = \frac{1}{1 + T_i s} $$ (10.23)

$$ F_s = F_o $$ (10.24)

and

$$ \Omega = \omega_{\delta} - \omega_{m} $$ (10.22)

To proceed further with the calculations we make following assumptions: (i) the high frequency gain \( F_o \) of the filter is much less than unity, i.e., \( T_o \gg T_i \) and (ii) \( T_i \) is itself in milliseconds. We have seen in Chapter 7 that for a first order PLL operating in the steady state condition, the phase-detector output consists of a dc voltage along with ac components (cf. 7.56). Thus when a filter network of the form of (10.20) is connected across the output of the phase detector, a slowly varying voltage will develop across the filter capacitor. Now if the time constant \( T_i \) is large, the voltage developed across the capacitor will not change appreciably over the time intervals of the order of the beat note period \( T_o \). The term \( A K F_s \sin \phi \) gives a measure of this slowly varying dc voltage across the filter capacitance. Therefore, over the intervals of the order of the beat note period, (10.22) looks like the equation of a first order PLL with average detuning

$$ \bar{\delta} = \Omega - \langle \delta \rangle $$ (10.29)
and a loop gain of $AKF_0$.

where $\zeta > 0$ presents the average over one cycle of the beat note. Since the voltage that helps the loop to pull-in, is a slowly varying function of time and the rate of change of the beat note frequency is small compared to the beat note frequency, we write

$$\text{AKF}_0 \sin \varphi = \frac{1}{T_0} \int_{0}^{T_0} \text{AKL}(t) \sin \varphi \, dt$$

$$= \frac{\text{AKL}(t) - E_0}{1 + pT_1} \sin \varphi \, dt$$

Again, noting that the equivalent first order loop has a loop gain of $\text{AKF}_0(\text{AKF}_0)$ and a frequency error $\omega_0$, the average value of the phase detector output is given by (cf. 7.56)

$$\langle A \sin \varphi \rangle = A(\omega_0 - \sqrt{\omega_1^2 - (\text{AKF}_0)^2 \omega_0}) \tag{10.27}$$

Therefore, computing (10.25), (10.26) and (10.27) one finds that

$$\omega_1 = \Omega + \frac{1 - E_0}{1 + pT_1} \sqrt{\omega_1^2 - (\text{AKF}_0)^2 \omega_0} \tag{10.28}$$

Now putting

$$x = \frac{\omega_0}{\text{AKF}_0} \text{ and } x_0 = \frac{\Omega}{\text{AKF}_0}$$

one can form a differential equation [6] in $x$ at

$$T_1 \frac{dx}{dt} = x - x_0 + \frac{1 - E_0}{E_0} (\sqrt{\omega_1^2 - \omega_0} - \omega_1) \tag{10.30}$$

i.e.,

$$T_1 \frac{dx}{dt} = x - x_0 + \frac{1 - E_0}{E_0} (\sqrt{\omega_1^2 - \omega_0} - \omega_1) \tag{10.30a}$$

To find the frequency acquisition time, the above equation is to be integrated between specified limits. The lower limit of the integration is obviously $x_0$. To find the upper limit we remember the equivalent first order loop with the detuning $\omega_0$ and loop gain $\text{AKF}_0$. Obviously, the beat note frequency tends to zero as $\omega_0$ approaches $\text{AKF}_0$, i.e., $x$ becomes unity. Thus the frequency acquisition time $T_f$ is given by
Now putting
\[ y = \sqrt{x^2 - 1} - x \]
\[ x = \frac{1 + y^2}{2y} \text{ and } \frac{dx}{dy} = -\frac{1}{2} (1 - y^2) \]
Thus (10.31) can be written as
\[ T_F = T_0 \int_0^1 \frac{(y - 1)y}{(2 - F_0) y^2 + 2 F_0 x y + F_0} \frac{dy}{y} \]
(10.32)
The integration in (10.32) can be carried out \([6]\). However, an expression for the locking range can be deduced from (10.35) by remembering that the locking range would be that value of \( \Omega \) i.e.,
\[ \frac{\Omega}{\Delta K} = F_0 x_L \]
for which \( T_F \) is infinite. This happens when the denominator of the integrand has a real root; hence the loop will lock if
\[ \frac{\Omega}{\Delta K} \ll 2 F_0 - F_0^2 \]
(10.33)
A simple expression for the frequency pulling time \( T_F \) can be obtained from (10.31) for the case when \( F_0 \ll 1 \). Thus, in this situation, (10.31) reduces to
\[ T_F \approx T_0 \int_0^1 \frac{F_0 dx}{(1 - F_0) (\sqrt{x^2 - 1} - x)} \]
\[ = - F_0 T_0 \int_0^1 \frac{dx}{(\sqrt{x^2 - 1} + x)} \]
\[ = \frac{4 F_0 T_0 \sqrt{x_L^2 - 1} + \ln(x_L + \sqrt{x_L^2 - 1})}{(1 - F_0) / \Delta K} \]
(10.34)
which, for large values of \( x_L = \Omega / \Delta K F_0 \), reduces to
\[ T_F \approx T_0 \frac{\Omega^2}{\Delta K} - 4 F_0 T_0 \ln \left( \frac{\Omega}{\Delta K F_0} \right) \]
(10.35)
This expression is similar to that later developed by Veizerki (cf.
10.14) except for the second term of (10.33) which Richman also neglected. Variation of the frequency pulling time $T_p$, as computed from (16.32) via a digital computer, is shown in Fig. 10.5.

Fig. 10.5. Variation of the frequency pull-in time with the locking ratio.

10.3. Approximate Acquisition Analysis of the PLL Incorporating an Imperfect Integrator

In this we consider the analysis of a PLL incorporating a filter network with transfer function.
\[ F(t) = \frac{1}{1 + sT} \]  

(10.36)

Normally the value of the time constant is large. The loop equation in this case is given by,

\[ T \frac{d\varphi}{dt} + \frac{d\varphi}{dt} + AK \sin \varphi = V \]  

(10.37)

Before we start analyzing this loop, let us remember the characteristics of a first order PLL in the steady state condition, i.e., when \( S \gg AK \). We have noted that the output of the phase detector consists of a dc component and a large number of harmonic components, the frequencies of which are multiple of the best angular frequency (\( \sqrt{V^2 - AK^2} \)). Now when a low pass filter with large time constant (cf. 10.36) is incorporated in the loop, it may be assumed that the phase detector output and hence the phase modulation of VCO occurs at the fundamental best angular frequency and the harmonic components are suppressed by the loop filter.

Thus, referring to (7.50), we now write

\[ \varphi = \omega t + a + m \sin (\omega t - \beta) \]  

(10.38)

where \( \omega \) is the best angular frequency and \( a, m, \beta \) are constants. Further, we have noted that during the process of pull-in the best angular frequency gradually changes due to the gradual change of the dc output of the phase detector. Thus during acquisition we rewrite (10.38) as

\[ \varphi = \int \omega(t) \, dt + a + m(t) \sin \left( \int \omega(t) \, dt - \beta \right) \]  

(10.39)

where \( \omega(t) \) and \( m(t) \) are slowly varying function of time. Finally,

\[ \psi = \int \omega(t) \, dt + a \]  

and

\[ \gamma = \omega t + \beta \]

one finds

\[ \frac{d\psi}{dt} = \omega + \omega_0 \cos (\psi - \gamma) + \frac{d\psi}{dt} \sin (\psi - \gamma) \]  

(10.40)

\[ \frac{d\varphi}{dt} = \omega + \frac{d\varphi}{dt} \sin (\psi - \gamma) - \omega m \sin (\psi - \gamma) \]  

(10.41)
Note that in the above $\omega$ and $m$ are functions of time. Further in view of (10.39) one finds that

$$\sin \varphi = \frac{m}{2} \sin \Theta - \frac{m}{2} \sin \left( \Theta - 2\gamma \right) - \frac{m}{2} \sin \gamma$$  \hspace{1cm} (10.42)

where $J_0(x)$ is the Bessel's function of order 0 and argument $x$. Referring to (10.37), (10.40), (10.41), (10.43) and equating the coefficients of $\sin \Theta$ and $\cos \Theta$ we find

$$\frac{d}{dt} \left( \frac{\omega m}{2} \right) = -\frac{\omega m}{2} \frac{\partial J_0}{\partial \Phi}$$  \hspace{1cm} (10.45)

Remembering that $\frac{d}{dt}$ and $\frac{dm}{dt}$ are small, one can find

$$\tan \gamma \cos \gamma = -\frac{1}{\omega T}$$  \hspace{1cm} (10.46)

since $m$ is also small, one gets

$$m = \frac{AK}{\omega^2} \frac{1}{\cos \gamma}$$  \hspace{1cm} (10.47)

Therefore, using (10.36), (10.47) and (10.43) and putting

$$J_0(m) \approx \frac{m}{2}$$  \hspace{1cm} (10.48)

one gets

$$\frac{d}{dt} \left( \frac{\omega m}{2} \right) + \frac{\omega m}{2} \frac{\partial J_0}{\partial \Phi} = \Omega$$  \hspace{1cm} (10.49)

Now assuming $\omega T \gg 1$, we get

$$\frac{d}{dt} \left( \frac{\omega m}{2} \right) + \frac{0.5mK}{\omega^2 T^2} = \Omega$$  \hspace{1cm} (10.50)

Therefore,

$$\frac{dt}{\Omega - \omega} = \frac{TmK}{0.5m^2}$$
To find the frequency acquisition range the above equation has to be integrated between specified limits. The lower limit of the integration is obviously \( \omega_i \), the initial detuning. The upper limit of (10.50) is taken to be \( \frac{1}{\sqrt{\gamma}} \), the value of beat angular frequency after which the loop does not slip cycles. Thus the frequency acquisition time is given by

\[
\tau_Y = \frac{\sqrt{\gamma}}{\Omega - \omega_i - \frac{0.5AK^2}{\omega^2Y}}
\]

(10.51)

which, after change of variables, can be written as

\[
\tau_Y = \frac{\sqrt{\gamma}}{10AK^2} \left( x^2 - \sqrt{AK^2(\Omega AK)x^2 + 0.5} \right)
\]

(10.52)

To calculate the locking range we remember that it is that value of initial detuning for the frequency acquisition time to be infinite. This happens when the denominator of the integrand (10.52) has a real root. That is, when

\[
\frac{\Omega}{AK} = \frac{1.475}{\sqrt{AK^2}}
\]

(10.53)

Note that the above expression is the locking range of a PLL, incorporating an imperfect integrator, when the values of \( AK^2 \) is large. If agrees well with the experimental results of Ray [2, 3] for \( AK^2 \) larger than 10.0. This result also tallies well with that of Stockham [3] (see ref. 15 of chapter 15). Variation of the locking range with the filter time constant is shown in Fig. 10.6. Variation of the frequency acquisition time is also depicted in Fig. 10.7.

10.4 Approximate Acquisition Analysis of a Second Oliver Type-2 Loop

In this section we consider a PLL with an integrating filter having transfer function

\[
F(s) = \frac{1 + sT_k}{2T_k}
\]

As such the loop equation is written as
Fig. 10.6. The locking range characteristic for an imperfect integrator.

\[
T_I \frac{d \phi}{dT_I} + AK T_I \frac{d \phi}{dT_I} \sin \phi + AK \sin \phi = 0, \quad (10.54)
\]

As discussed in section 10.3, we assume that \( \gamma \) is given by (10.39) and proceeding in an analogous way, one can show that

\[
T_I \frac{d m}{dT_I} + AK T_I \sin \gamma \frac{d \gamma}{dT_I} = AK \gamma (m) \sin \gamma \quad (10.55)
\]

\[
T_I \frac{d m}{dT_I} (m) + u T_I \frac{d m}{dT_I} + AK T_I \nu (\lambda(m) - J_{\gamma}(m)) \cos \gamma
\]

\[
+ AK (\lambda(m) + J_{\gamma}(m)) \sin \gamma = 0 \quad (10.56)
\]

and

\[
AK (\lambda(m) - J_{\gamma}(m)) \cos \gamma - AK \nu T_I (\lambda(m)) \sin \gamma
\]

\[
= u^3 T_I \gamma \quad (10.57)
\]

Further, assuming that \( \omega(t) \) and \( m(t) \) are slowly varying functions of time, and noting that as \( m \) is small, one finds that

\[
\tan \gamma = - \frac{\gamma}{\omega T_I}
\]

\[
f_T(m) = \frac{m}{2} \quad (10.58)
\]

and

\[
m = \frac{AK T_I}{\omega T_I} \quad (10.59)
\]
Fig. 10.7. The locking time characteristics of a PLL with imperfect integrator.

Putting these values in (10.53) it is easily shown that

\[ T_{in} \text{in} + \frac{A^2K_T}{2T_3} \text{in}^2 + \frac{A^2K_T}{2T_1} \text{in} \cdot \text{in} = - \frac{A^2K_T}{2T_4} \text{in} \]  \hspace{1cm} (10.59)

Putting

\[ 2T_4 \text{in} = \frac{AK_T}{T_3} \]
278 Phase Lock Theory and Applications

one finds that: (10.59) reduces to

$$\left( \frac{aT_1}{24I_a} + \frac{T_2}{a} \right) w_n = -v$$  \(\text{(10.60)}\)

Defining the frequency acquisition time to be the time necessary for reducing the frequency error from $\Omega$ to $2\Delta$ $w_n$, one finds, by integrating (10.60) between these limits, that

$$T_p = \frac{T_2}{24I_a} \left( \frac{\Delta^2}{4I_a^2} + T_1 \ln \left( \frac{\Omega}{2\Delta} \right) \right)$$ \(\text{(10.61)}\)

We have seen that this type of loop has an infinite locking range. Further if $\Omega$ is large compared to $2\Delta$, $w_n$, (10.61) can be written as

$$T_p \approx T_1 \left( \frac{\Omega}{2\Delta} \right)^\alpha$$ \(\text{(10.62)}\)

This is similar to the Velerbi's result (cf. 10.18).

10.5 Unified Approach for Acquisition Analysis of a PLL Incorporating Filter Networks with and without High Frequency Gain

We have already seen that different analytical approaches are required for finding the frequency acquisition time and the locking range for PLL's incorporating filter networks with and without finite high frequency gain. For example, Richman's approach does not apply to the case where the h.f gain $F_0 = 0$. Similarly, the analysis of the section 10.3 are restricted to the case, when the filter network of the PLL does not have h.f gain.

Let us consider a PLL with such filters and thus the loop equation can be written as

$$\frac{d\varphi}{dt} = \Omega - AKF(p) \sin \varphi$$ \(\text{(10.63)}\)

In the beating condition, we now assume the following solution for $\varphi$:

$$\varphi = \int a(t) dt + a(0) + \int m(t) \sin \varphi dt$$ \(\text{(10.64)}\)

The time varying nature of $a(t)$ and $m(t)$ ensures that the system gradually can approach the locked state from the unlocked state by a slow increase of the phase detector output voltage (14).
Thus we have
\[ \sin \varphi = \sin (F \tau + s) \cos (m \sin \varphi) + \cos (F \tau + s) \sin (m \sin \varphi) \]
\[ = j_1 (m) \sin (F \tau + s) - j_0 (m) \sin (F \tau - s) - j_1 (m) \sin \alpha \]

Using (10.67) and (10.69) one easily finds that
\[
\frac{d \psi}{dt} = \omega - AKF (\omega) j_1 (m) \sin \alpha = 0 \tag{10.66}
\]
\[
\frac{dn}{dt} = -AKF (\omega) j_1 (m) \cos (-\varphi (m))
\]
\[ - j_0 (m) \cos (\varphi + \varphi (m)) \tag{10.67}
\]
\[\mu = -AKF (\omega) j_1 (m) \sin (\varphi - \varphi (m)) + j_0 (m) \sin (\varphi + \varphi (m)) \tag{10.68}\]

where
\[ F (m) = F (\omega) \exp (-j \varphi (m)) \tag{10.69}\]

Since \( m \) is small and also varies slowly with time, one finds from (10.67) that
\[ a \approx \varphi (m) \approx \pi / 2 \tag{10.70} \]

Referring to (10.64) one notes that in the stationary state i.e., when
\[ \frac{d \psi}{dt} = \frac{d^2 \psi}{dt^2} = 0, \]

is approximated by \( n = 1, s = 0 \) or by \( n = -1, s = 0 \). Thus we assume that during looking \( m \) varies but remains close to unity during the process of frequency acquisition, i.e., when the beat angular frequency changes from \( \Omega \) to \( AKF \). Therefore, the average value of the beat angular frequency during pull-in is obtained from (10.66) and (10.70)
\[ \langle \omega \rangle = AKG (\omega) j_1 (m) \approx 0.75 AKG (\omega) \tag{10.71}\]

where
\[ F (\omega) = \frac{1}{\Omega - AKF} a^{-\alpha} \int_0^{\alpha} F (\omega) \, ds \tag{10.72}\]

Now, referring to (10.66), one finds that the average change in \( \frac{d \psi}{dt} \) is given by \( n = -1 \).
\[
\frac{d\phi}{dt} = \Omega - \langle \phi \rangle - \langle \zeta \rangle, \text{ (m) } AKF(0) \sin \phi
\]

i.e.,
\[
\frac{d\phi}{dt} = \Omega - 0.75AK \langle F(\phi) \rangle - 0.75 AKF(0) \sin \phi
\]

(10.73)

This is approximated on the assumption that \( m \) remains close to unity and \( \langle \sin \phi \rangle \) is represented by a periodic triangular wave of maximum value of unity at \( x = \pi/2 \) and then reverting back to \( \sin \phi \). Again referring to (10.65) and (10.64) one finds that in the locked state:
\[
\sin \langle \phi \rangle = \frac{\Omega}{AKF(0)}
\]

(10.74)

Therefore, from (10.73) and (10.74), one finds that the locking range of the PLL is given by
\[
\Omega = AK \langle F(\phi) \rangle
\]

(18.75)

where \( \langle F(\phi) \rangle \) is given by (10.72).

The above equation can be graphically solved to find the values of the locking range \( \frac{\Omega}{AK} \). This is illustrated in Fig. 10.8 and Fig. 10.9 for the filter networks having transfer functions respectively.

![Fig. 10.1. Illustrating graphical solution for the locking range of a PLL with](image_url)
Fig. 10.9. Illustrative graphical method for evaluating the locking range of a PLL with $\tilde{R}(\nu) = \frac{1 + P_1 \nu F}{1 + \nu^2}$.

$$\tilde{R}(\nu) = \frac{1}{1 + \nu^2}$$

and

$$F(\nu) = \frac{1 + \nu^2 F_1}{1 + \nu^2 F_2}$$

Frequency acquisition time can be found from (10.72) by integrating it over the limits of initial and final values of $\Omega$. Thus

$$T_F = \frac{\nu}{\Omega_0 - \Delta K_2 \sin \alpha}$$

(10.76)

where

$$\Omega_0 = \Omega - 0.75 AK \langle P(\alpha) \rangle$$

(10.77)

$$AK_2 = 0.25 AK_{P(\alpha)}$$

(10.78)

$$\alpha_0 = \alpha_0 / 2 + \phi(\Omega)$$

(10.79)

Variations of locking range with $AKT$ for different values of $\Delta K_2$ are shown in Fig. 10.10. It is seen that for large values of the time-constant of the filters, Richman's theory fits well with the computer-simulated results.
10.6 Locking Characteristics of a PLL with Triangular, Sawtooth and Rectangular Type of Phase Detectors

In this section we discuss a simple method of calculating the locking range and pull-in time of a PLL incorporating an arbitrary periodic phase detector \([8, 15]\) and a filter network with finite highfrequency gain. We mainly follow the method of Venners and Starner. Let us consider the filter network of having the transfer function

\[
F(s) = F_s + \frac{1 - F_s}{1 + jTF_s}
\]

We rewrite (10.22) replacing \(\phi\) by \(S(\phi)\). Thus

\[
\frac{d\phi}{dt} = \left(\Omega - AKF_t(\phi)\right) - AKF_t(\phi)
\]  

(10.80)

Here \(g(\phi)\) is a periodic function of \(\phi\) with module-2\(\pi\). Following the approximations and arguments of section 10.2. We write the following expression for the average detuning

\[
\bar{\phi}_d = \Omega - \langle AKF_t(\phi)\rangle
\]  

(10.81)

We have seen that substitution of this in the original governing equation of the PLL, reduces a second order PLL to an equivalent first order PLL. Note that (cf. 10.26)
\[ \langle \Delta F \Delta s \rangle = \frac{AK(1 - F_{0})}{1 + F_{0}^{2}} \int_{0}^{T_{s}} g(\phi) \, d\phi \]  

(10.82)

Now comparing (10.80) and (10.81) one can approximately write

\[ \frac{d\phi}{dt} \approx \tilde{\omega}_{0} - AKF_{0} g(\phi) \]  

(10.83)

Thus using this, one gets for

\[ \langle g(\phi) \rangle = \frac{1}{T_{s}} \int_{0}^{T_{s}} g(\phi) \, d\phi \]

\[ = \frac{1}{\tilde{\omega}_{0}} \int_{0}^{T_{s}} g(\phi) \, d\phi \]

(10.84)

which for a filter network of negligible h.f. gain can be approximated by

\[ \langle g(\phi) \rangle \approx \frac{1}{\tilde{\omega}_{0} T_{s}} \int_{0}^{2\pi/\tilde{\omega}_{0}} \left( 1 + \frac{AKF_{0}}{\tilde{\omega}_{0}} g(\phi) \right) \, d\phi \]  

(10.85)

since \( \tilde{\omega}_{0} \) is varying very slowly, it may be assumed to be constant over a beat period. If the phase detector is a triangular one \( g(\phi) \) or a sawtooth one \( g(\phi) \), with a maximum value of unity, as given by

\[ g(\phi) = \begin{cases} \frac{2}{\pi} \phi, & 0 < \phi < \pi/2 \\ 2, & \pi/2 < \phi \leq 3\pi/2 \\ -2, & \pi < \phi < 2\pi \end{cases} \]  

(10.86)

or

\[ g(\phi) = \begin{cases} \frac{2}{\pi} \phi, & 0 < \phi < \pi/2 \\ 2, & \pi/2 < \phi \leq \pi \end{cases} \]  

(10.87)

In these two cases, the beat period for a first order PLL is

\[ T_{b} = \frac{\pi}{AK} \log \frac{1 + AK\Omega}{1 - AK\Omega} \]  

(10.88)

and

\[ T_{b} = \frac{\pi}{AK} \log \frac{1 + AK\Omega}{1 - AK\Omega} \]  

(10.89)

and \( \langle \phi(\eta) \rangle \) for these cases are
\[ \langle \Phi \rangle = \frac{1}{T_s} \int_{t_0}^{t_0 + T_s} \Phi(t) \, dt \]

\[ = \int_{t_0}^{t_0 + T_s} \Phi(t) \, dt \]

\[ = \frac{1}{T_s} \int_{t_0}^{t_0 + T_s} \Phi(t) \, dt \]

\[ = \frac{1}{T_s} \int_{t_0}^{t_0 + T_s} \Omega \, d\Phi \]

\[ = \frac{\Omega}{AK_s} \frac{2}{\log \left( \frac{1 + AE/\Delta}{1 - AE/\Delta} \right)} \]  \hspace{1cm} \text{(10.90)}

Therefore, comparing this with (10.5d) and noting that \( F_s = F_0 \), one finds

\[ \Delta \Phi(\Omega) = A \left( \frac{x - 2}{\log \left( \frac{x + 1}{x - 1} \right)} \right) \]  \hspace{1cm} \text{(10.91)}

where \( x = \frac{\Phi}{AK_s} \).

Therefore, from (10.61) one finds that

\[ \Delta \Phi = \Omega \frac{AE(1 - F_s)}{1 + F_s^2} \langle \Phi \rangle \]

\[ - \Omega \frac{AE(1 - F_s)}{1 + F_s^2} \left[ \frac{2}{\log \left( \frac{x + 1}{x - 1} \right)} - x \right] \]

\[ = \Omega \frac{AE(1 - F_s)}{1 + F_s^2} \left[ \frac{2}{\log \left( \frac{x + 1}{x - 1} \right)} - x \right] \]

\[ = \frac{x}{AK_s} \frac{1 - F_s/2}{1 + F_s^2} \left[ \frac{2}{\log \left( \frac{x + 1}{x - 1} \right)} - x \right] \]  \hspace{1cm} \text{(10.92)}

where

\[ x = \frac{\Phi}{AK_s} \text{ and } \frac{\Phi}{AK_s} = \frac{\Omega}{AK_s} \]

\[ \frac{\Phi}{AK_s} = x + \frac{1 - F_s}{F_s} \left[ \frac{2}{\log \left( \frac{x + 1}{x - 1} \right)} - x \right] \]

Therefore, the frequency pulling time is given by (cf. 10.31)
\[ T_y = T_y \left[ x_0 - x + \frac{1}{2} \ln \left( \frac{x}{x_0} \right) \right] \]

(for \( x = 1.2 \)) \( (10.93) \)

Approximating \( 2 \log \left( \frac{x + 1}{x - 1} \right) \) by \( x(x - 1)/(x + 1) \) one can rewrite (remembering that \( 2 \log \left( \frac{x + 1}{x - 1} \right) \) is valid for \( x > 1 \)) \( (10.93) \) as

\[ T_y = T_y \left[ x_0 - x + \frac{1}{2} \frac{F_0}{F_y} (x - x_0) \right] \]

\[ = -T_y \left[ x_0 - \left( 1 + \frac{x_0}{x} - \frac{0.91F_0}{x_0} \right) x + 0.91x_0 \right] \]

(10.94)

By putting

\[ 2b = 1 + x_0 - 0.91F_0 \]

\[ C = 0.91x_0 \]

We rewrite \( (10.94) \) as

\[ T_y = -T_y \int \frac{x}{x - 2bx + C} \, dx \]

(10.95)

The solution of \( (10.95) \) depends on whether \( x^2 - 2bx + C = 0 \) has real roots or not, and so we consider the following two cases.

Case 1: \( b > C \)

In this case both the roots of \( x^2 - 2bx + C = 0 \) are real and are given by

\[ x_1, x_2 = b \pm \sqrt{b^2 - C} \]

Thus the solution of \( (10.95) \) can be written as

\[ T_y = -T_y \left[ \log \left( x^2 - 2bx + C \right) \right]_{x=x_1}^{x=x_2} \]

(10.96)
To see whether the system locks or not in this case we rewrite the integration for any time \( t \) as

\[
\begin{align*}
\int_{t_0}^{t} &= -T_x \left[ \frac{1}{2} \log_b \left( x^2 - 2bx + C \right) \\
&+ \frac{1}{2} \left( b - C \right) x \right] \\
&\quad + \frac{1}{2} \left( b - C \right) x \\
&= -T_x \left[ \frac{1}{2} \log_b \left( x^2 - 2bx + C \right) \\
&+ \frac{1}{2} \left( b - C \right) x \right] \\
&\quad + \frac{1}{2} \left( b - C \right) x
\end{align*}
\]  
(10.96)

where \( T_x \) is an integration constant. In this case one finds from (10.36) that when \( t \) tends to infinity, \( x \) reaches a finite value. This indicates that PLL does not lock when \( b' > C \).

Case II: \( b' < C \)

Following the same sort of analysis as that of case I, one finds that the system locks in this case. The locking time is given by

\[
T_x = T_x \left[ -\frac{1}{2} \log_b \left( x^2 - 2bx + C \right) \\
&+ \frac{1}{2} \left( b - C \right) x \right] \\
&\quad + \frac{1}{2} \left( b - C \right) x
\]  
(10.97)

Thus the limiting value of \( x \) and hence the locking range is found when

\[
b' = C
\]

i.e.,

\[
(1 + x_0 = 09/F_0)^2 = 3.64x_0
\]  
(10.98)

Hence the locking range is found by solving (10.98) for \( x_0 \)

\[
\frac{Q}{AK} = (3.23 + 0.9) = 0.5724 \sqrt{F_0 / (1 - F_0)}
\]  
(10.99)

This result agrees well with that of Kharanov [7] for all values of \( F_0 \) except close to unity.

For a rectangular type of phase detector we can follow a similar procedure, by noting that

\[
g(\eta) = 1 \quad 0 \leq \eta \leq \pi
\]

\[
g(\eta) = -1 \quad -\pi \leq \eta \leq 0
\]

Now for a first order PLL, incorporating this type of phase detector, one gets

\[
\frac{d\eta}{dt} = \Omega - AK
\]  
for \( 0 \leq \eta \leq \pi \)
and
\[
\frac{dp}{df} = \Omega + AK
\]
for \( -\pi \leq \phi \leq 0 \)

Therefore, when \( \Omega > AK \), i.e., out of lock, the best period is given by
\[
(T_\text{best}) = \frac{2\Omega}{\Omega^2 - AK^2}.
\]
(10.100)

This is calculated by noting the time taken to change the phase difference from \(-\pi/2\) to \(\pi\).

Thus, using (10.55), the average value of \(g(\phi)\) is easily shown to be given by
\[
\langle g(\phi) \rangle = AK\Omega^2.
\]
(10.101)

Hence, referring to (20.81) one finds that
\[
x = x_0 - \frac{1 - F_n}{1 + \frac{1}{2}F_n} x
\]
(10.102)

Before proceeding further with the calculation it is important to note that the above relation (10.102) is true for small values of \(F_n\).

Now from (10.102) one gets
\[
\frac{dx}{dt} = -T_1 \frac{i}{x} - x^2 + \frac{1}{2}x^2 + (1/F_n - 1)
\]
(10.103)

From which it is seen that the loop will lock provided
\[
\frac{\Omega}{AK} < 2\sqrt{F_n(1 - F_n)}
\]

Although this expression is similar to that derived by Zhukov, (following the procedure of Krapner), who claims to be valid for all values of \(F_n\). But it is valid for small values of \(F_n\). For example, if \( F_n = 1 \), the locking ratio becomes zero. That is, pulling range is given by
\[
\frac{\Omega}{AK} = 2\sqrt{F_n(1 - F_n)}
\]
for \( 0 \leq F_n \leq 1 \)
(10.104)

The frequency pulling time is obtained from (10.102) as
\[
T_p = \frac{1}{F_n} \int \frac{x dx}{\left(x^2 - xK + (1/F_n - 1)\right)}
\]
which is similar to the force of (10.92). Hence one finds that
Phase Lock Theories and Applications

\[ T_1 = T_0 \left[ -\frac{1}{2} \log_e \left( x^2 - x_0 + 1/F_e - 1 \right) \right] \]

\[ = \frac{x_1}{1+F_e - 1 - x_0^2/4} \sqrt{1/F_e - 1} - x_0^2/4 \tan^{-1} \frac{x - x_0/2}{\sqrt{1/F_e - 1} - x_0^2/4} \]  

(10.103)

10.7 Simplified Formula for an Arbitrary Periodic Phase Detector

The philosophy of evolving a simple formula for calculating the locking range of a PLL with an arbitrary periodic phase detector depends on replacing the actual phase detector by an equivalent sinusoidal phase detector, such that the average phase detector output of the equivalent phase detector is equal to that of the actual phase detector.

Consider a PLL with an arbitrary phase detector which is an odd periodic function of the phase difference. Thus we replace the following phase governing equation of the actual PLL

\[ \frac{dp}{dt} = \Omega - AKF \phi \]  

(10.106)

by the equivalent equation

\[ \frac{dp}{dt} = \Omega - AKF \phi \sin \phi \]  

(10.107)

so far as the pulling behaviour is concerned. We have already seen that the pulling behaviour depends on the average phase detector output voltage. Thus for a filter network of the form

\[ F(t) = F_e + \frac{1}{1 + T_s^2} F_0 \]

We rewrite (10.106) and (10.107) as

\[ \frac{dp}{dt} = \left( \Omega - AKF \phi \sin \phi \right) - AKF \phi \]  

(10.108)

and

\[ \frac{dp}{dt} = \left( \Omega - AKF \phi \sin \phi \right) - AKF_\phi \sin \phi \]  

(10.109)

For DC output of the arbitrary phase detector for the equivalent first order loop

DC output of the arbitrary phase detector for the equivalent first order loop
$\langle \phi(t) \rangle = x_1 \left( \frac{2\pi}{A\Delta f_{o}} \right)^{2} x_2 \left( \frac{2\pi}{A\Delta f_{n}} \right)^{2}$

(10.110)

Now if $P_{i}$ is small, we know that (cf. 10.85)

$\langle g(t) \rangle \approx \frac{1}{\omega_{T} P_{i}} \left( \frac{4KE_{n}}{\omega_{i}} \right)^{2} \int_{0}^{\pi} g^{2}(\theta) d\theta$

(10.111)

i.e.,

$\langle g(t) \rangle \approx \frac{4KE_{n}}{\omega_{i}} \frac{1}{2\pi} \int_{0}^{\pi} g^{2}(\theta) d\theta$

(10.112)

Further for small values of $P_{i}$ the right hand side of (10.110) can be approximated by

$\frac{x_1}{2} \left( \frac{2\pi}{A\Delta f_{o}} \right)^{2}$

Thus using (10.110) and (10.112) one finds

$x_1 = \sqrt{2} g^{2}(\theta)$

(10.113)

where

$\langle g(t) \rangle = \frac{1}{2\pi} \int_{0}^{\pi} g^{2}(\theta) d\theta$

(10.114)

Therefore, so far as the pull-in behaviour of a PLL is concerned, our previous formulae for pull-in range and lock-in time can be applied to a PLL incorporating an arbitrary periodic phase detector provided we replace the output of the phase detector by $\Delta V \sqrt{2} g^{2}(\theta)$, sin $\theta$ in place of $A$ sin $\theta$.

REFERENCES

3. Al Viterbi, "Acquisition and tracking behaviour of phase locked loops", 
250 Phase Lock Theories and Applications

It is clear from the study of the previous chapter that a phase-lock-
loop of choice should have the following properties:

1) High noise immunity, that is, the noise bandwidth should be as
low as possible.

2) Large locking range which indicates the capability of a PLL
to track an unknown signal with large frequency offset.

3) Small locking time, that is, the time a loop takes to acquire the
unknown signal.

Large phase locking range, i.e., the loop allows larger phase

difference than $\pi$.

We have not yet discussed anything about the fourth one. It is the
maximum permissible value of the phase difference between the local
oscillation and the synchronizing signal up to which synchronism
between them could be maintained. Obviously, the phase locking
range of a simple PLL is $\pi/2$, i.e., the maximum value of

$\phi = \frac{\pi}{2}$. If we have seen that as soon as the phase difference

exceeds $\pi/2$, the loop slips cycles. To appreciate the reason for having

a large phase locking range (PLR), let us consider the situation when

the signal is accompanied by noise. Because of the random nature

of the unwanted signal, the effective phase difference between the

incoming signal and the local oscillation at times will exceed $\pi/2$ and

thus the loop will slip cycle in following the incoming signal. This

will lead to loss of information. Obviously, the larger the value of

PLR better is the performance of the loop.

Referring to the expressions for the noise bandwidth (cf. 9.53, 9.67, 9.71), the locking range (cf. 10.13, 10.35, 10.73) and the locking
time (cf. 10.14, 10.18, 10.35), it is easily seen that a change of one
of these is the favourable direction leads to the deterioration of the other two. In the sections to follow, we will now consider a few modified loops, that achieve the stipulated purpose to some extent.

11.1 Extended Range Phase Locked Loop (1, 2)

A simple arrangement of the extended range PLL is shown in Fig. 11.1. It consists of a phase detector, a low pass filter, a VCO and a phase modulator. Assume that the incoming signal and the output of the phase modulator are respectively of the forms

\[ \nu(t) = \sqrt{2}A \sin (\omega_0 t + \Psi_f(t)) \]  \hspace{1cm} (11.1)

and

\[ \nu(t) = \sqrt{2}K_c \cos (\omega_0 t + \Psi_v(t)) \]  \hspace{1cm} (11.2)

Before deriving the phase governing equation of the system let us note that \( \Psi_f(t) \) has two components, viz., (1) \( \Psi_d(t) \) the direct phase modulation of the VCO by the phase detector output through reactance modulation, and (2) \( \Psi_{pd}(t) \), the phase modulation of the VCO output through the phase modulator or the electronic phase shifter. Putting

\[ \Psi = (\omega_0 - \omega_0) t + \Psi_d - \Psi_{pd} \]  \hspace{1cm} (11.3)

one finds that the phase detector output is given by

\[ \nu_f = AK_c K_v \sin \Psi \]  \hspace{1cm} (11.4)

Noting that

\[ \Psi_d(t) = \Psi_d(t) + \Psi_{pd}(t) \]  \hspace{1cm} (11.5)
one finds that
\[ V_{\text{mod}}(t) = \frac{K_m}{p} (F_p(p) \Delta K K_2) \sin \psi \]  
(11.6)
and
\[ V_{\text{mod}}(t) = \frac{K_m}{p} (F_p(p) \Delta K K_3 \sin \psi) \]  
(11.7)
where \( K_m \) is the sensitivity of the phase modulator. Thus one finds that the phase equation of the ERPLL is given by
\[ \frac{d\psi}{dt} = \Omega - AK_1 K_2 K_3 (p \psi + p K_0) \sin \psi \]  
(11.8)
i.e.,
\[ \frac{d\psi}{dt} = \Omega - AK_1 K_2 K_3 (1 + p T_m) \sin \psi \]  
(11.9)
where
\[ \Omega = \omega_0 - \omega_n \]
\[ T_m = \frac{K_m}{K_3} \]
\[ K = K_1 K_2 K_3 \]
Equation (11.9) completely defines the locking behaviour of the loop. Putting \( \frac{d\psi}{dt} = 0 \) we can analyse the acquisition behaviour of the loop, following the procedure of Chapter 10. To study the noise filtering property we can follow the procedure as given in Chapter 9 and one can derive a similar equation to (9.34), viz.,
\[ \frac{d\psi}{dt} = \frac{d\psi}{dt} - K F (1 + p T_m) \sin \psi + \hat{N}(t) \]  
(11.10)
Note that \( F(t) \) is a bandlimited Gaussian noise. Note further that
\[ V_{\text{mod}}(t) = \frac{AF + P(p)}{p} \sin \psi \]
Moreover, if the strength of the incoming noise is not large, we may put \( \sin \psi \approx \psi \) in which case
\[ \psi(t) = \frac{\psi(t) - \epsilon \hat{g}(t) \epsilon}{\epsilon + \epsilon \hat{g}(t) \epsilon} \]  
(11.11)
where,
\[ \epsilon(t) = (1 + \epsilon T_m) \epsilon(t) \]  
(11.12)
Therefore, one finds that
Phase Lock Theory and Applications

\[ V_{out}(f) = \frac{AKF(f) \cdot V_T(f)}{s + AKQ(f)} + \frac{KE(f) \cdot N(f)}{s + AKQ(f)} \]  \hspace{1cm} (11.13)

Therefore, the noise bandwidth of the ERPLL, when the output is tapped from the VCO output, is given by (cf. 9.65)

\[ B_n = \frac{1}{4\pi} \int_{-\pi}^{\pi} |H(f)| \, df \]  \hspace{1cm} (11.14)

where

\[ H(f) = \frac{AKF(f)}{s + AK(1 + JT_n)F(f)} \]  \hspace{1cm} (11.15)

To judge the merit of the system let us, for example, consider a PLL with an imperfect integrator, i.e., \( F_1(f) = 1/(1 + sT) \). Assuming the time constant \( T \) to be large, the locking ratio of the PLL without the phase modulator is given by (cf. 10.75)

\[ \frac{\Omega}{\Delta K} = \frac{1}{\Omega} \int_{-\pi}^{\pi} |F_1(j\omega)| \, d\omega \]

where as that of the PLL with the phase modulator is given by

\[ \left( \frac{\Delta \omega}{\Delta K} \right) = \frac{1}{\Omega} \int_{-\pi}^{\pi} |(1 + j\omega T_n) F_1(j\omega)| \, d\omega \]  \hspace{1cm} (11.16)

where \( F_1 = \frac{\omega}{\Omega T} \)

If \( F_1 \) is small, Rimmich's formula can be used to calculate the lock-range. Thus it is obvious that \( (\Delta \omega/\Delta K) \) is less than \( (\Delta \omega/\Delta K)_0 \).

It is easy to show that noise bandwidths of the two systems are respectively given by

\[ B_n = \frac{\Delta \omega}{4} \]  \hspace{1cm} (11.17)

and

\[ (\Delta \omega)_0 = \frac{AK}{4} \cdot \frac{1}{1 + AKT_n} \]  \hspace{1cm} (11.18)

This shows that \((\Delta \omega)_0\) is less than \(B_n\).
11.2 Frequency Feedback Phase Locked Loop

This is a compound loop (3, 4) consisting of an automatic phase control circuit and an automatic frequency control circuit. A typical circuit configuration is shown in Fig. 11.2. Let the outputs of the VCO-1 and VCO-2 be represented by

\[
\begin{align*}
V_{\text{out}}(t) &= \sqrt{2} K_1 \cos (\omega_0 t + \varphi_1(t)) \quad (11.19) \\
V_{\text{out}}^*(t) &= \sqrt{2} K_2 \cos (\omega_0 t + \varphi_2(t)) \quad (11.20)
\end{align*}
\]

Assuming the input signal as given by (11.1), the outputs of the frequency discriminator and the phase detector are respectively given by

\[
\begin{align*}
\varphi &= \frac{1}{2 \pi} \int (\omega_1 - \omega_k - \omega_0) \, dt + \varphi_1(t) - \varphi_2(t) \quad (11.22) \\
\end{align*}
\]

and

\[
\varphi = \frac{1}{2 \pi} \int \omega_1 \, dt + \varphi_1 - \varphi_2 \quad (11.23)
\]

It is to be noted that the centre frequency of the discriminator,
having the sensitivity $a_d$ has been assumed to be the same as that of the VCO 2.

Now putting

$$\Omega = \omega_0 - \omega_f = \omega_{0d}$$

(11.24)

and assuming the sensitivities of the two VCO's to be identical ($K_d$)

we write,

$$\nu_{ad}(t) = \frac{AK_d}{2}(\omega_0 F_0(p) \sin \varphi)$$

(11.25)

and

$$\nu_{cf}(t) = \frac{K_d}{2}(\omega_0 F_0(p) \sin \varphi) + K_d \omega_f F_0(p) \sin \varphi$$

(11.25)

Therefore, comparing (11.23), (11.24), (11.25) and (11.26) we write,

$$\frac{d\varphi}{dt} = \frac{\Omega + \omega_0 F_0(p)}{1 + K_d \omega_f F_0(p) \sin \varphi + \frac{\omega_f F_0(p)}{1 + K_d \omega_f F_0(p)}}$$

(11.27)

where,

$$K_d = n_d K_d$$

(11.28)

Let us now derive the system equation when the incoming signal is accompanied by a narrow band gaussian noise, i.e., we write,

$$n(t) = \sqrt{2 \Delta} \sin (\omega_0 f + \nu(t)) + m(t)$$

(11.29)

where $n(t)$ is given by (9.13).

Rewrite $n(t)$ as

$$n(t) = \sqrt{2} N_d(t) \cos (\omega_0 f + \nu(t)) - \sqrt{2} N_d(t) \sin (\omega_0 f + \nu(t))$$

where

$$N_d(t) = n_d(t) \cos \nu(t) + n_d(t) \sin \nu(t)$$

and

$$N_d(t) = n_d(t) \cos \nu(t) - n_d(t) \sin \nu(t).$$

Assuming that the FFP LL is preceded by a bandwidth limiter, one can represent the net input to the mixer as

$$r(t) = \sqrt{2} \Delta \sin (\omega_0 f + \nu(t)) + \nu(t)$$

(11.30)

where it has been assumed that the strength of the noise is small compared to that of the carrier, and as such
\[ \mathcal{Y}_s = \arctan\left( \frac{N_s(t)}{A - \mathcal{K}G_0(t)} \right) \quad \text{(11.31)} \]

Therefore, in such a case, the system equation (11.27) will be the same except that \( \mathcal{Y}_s(t) \) is to be replaced by \( \mathcal{Y}_s + \mathcal{Y}_w \). Remembering this and taking the in-tune situation \( (\Omega = 0) \) one can write the following relation for a linear loop

\[ \phi(t) = -\frac{\mathcal{K}G_0(t)}{A + AG_0(t)} \mathcal{Y}_s(t) \quad \text{(11.32)} \]

Thus comparing (11.32) and (11.24) one finds that

\[ \mathcal{Y}_s(t) = \frac{AK_0(t) G_0(t) (\mathcal{Y}_s(t) + \mathcal{Y}_w(t))}{1 + AK_0(t) G_0(t)} \quad \text{(11.33)} \]

where

\[ G_0(t) = \frac{1}{1 + K_0 G_0(t)} \]

and

\[ G_1(t) = \frac{2 + K_0 G_0(t)}{1 + K_0 G_0(t)} \mathcal{Y}_w(t) \]

Therefore, the noise phase modulation of the VCO-2 is given by (cf. 11.23)

\[ \mathcal{Y}_s^2(t) = \frac{\mathcal{K}_0 G_0(t)}{2} N_s(t) \]

\[ = \mathcal{K}_0 G_0(t) N_s(t) \quad \text{(11.34)} \]

Hence, the noise bandwidth of the system, when observed at the output of the VCO-2, is given by

\[ B_n = \frac{1}{4\pi} \int_{-\infty}^{\infty} |H(f)|^2 df \quad \text{(11.35)} \]

Take a very simple situation such that bandwidth of the loop filter \( F(f) \) is large such that loop filtering is essentially done by

\[ F_0(f) = \frac{1}{l + j\pi f} \quad \text{(say)} \]

one finds,

\[ B_n = \frac{A}{\pi} \left( 1 + K_0 G_0(t) \right) \quad \text{(11.36)} \]

Further if one takes \( P_0(f) = \frac{1}{l + j\pi f} \), it can be shown that
\[ B_n = \frac{b_2 - a_2 b_3 / a_3}{a_4 (a_5 a_6 - 4 a_9)} \quad (11.37) \]

where,
\[ b_2 = A^2 k T_4, \quad b_3 = A^2 K_3 \]
\[ a_2 = T_2, \quad a_1 = T_1 (1 + K_2) + T_4 \]
\[ a_3 = 1 + K_2 + 2 T_4 A K \]
\[ a_4 = A R^2 + K_3 \]

11.2.1 SIGNAL ACQUISITION BY THE FFPLO

Narrow band loops are normally used for improving the noise performance of a PLL in the tracking mode. Performance is usually given to second order loops \( F(u) = (1 + A u)/ST_u \). For example, one may require a second order PLL with a noise bandwidth of 10Hz or so and damping factor of 0.90. In such a case using the formula for the acquisition time \( T_g = C P F^{1/2} B_\alpha \), one finds that it will be more an hour for the loop to acquire the signal for a frequency offset of about 1.0KHz. In order to reduce the acquisition time, various methods, a few of which will be described later, are used. For the moment, we will discuss how the FFPPLL can be used for improving the locking behaviour. For that matter let us consider the FFPPLL of Fig. 11.2, with a special type of frequency discriminator, the characteristic of which is shown in Fig. 11.3. That

![Diagram](image-url)

Fig. 11.3. The frequency discriminator characteristic with a dead zone.
is the characteristic has a dead-zone around zero frequency offset. The value of the dead-zone, \( \Omega_z \), is to be properly chosen depending upon the expected initial frequency offset of the incoming signal.

The loop operates in the following manner. When the initial frequency offset lies outside the dead-zone, the loop operates as a FPPLL. But as soon as the frequency error is sufficiently reduced so that the error falls within the zone \( \Omega_z \), the control voltage from the frequency discriminator is cut off and the loop operates in a simple PLL with the required bandwidth. The FPPLL in this case operates as a two-mode system.

Consider that the bandwidth of the filter network \( F(s) \) in the AFC loop is large, so that the operation of the FPPLL, when both the PLL and AFC circuits, is given by

\[
\frac{dy}{dt} = \frac{\Omega}{1 + K_y} \frac{2 + K_y}{1 + K_y} AK \frac{1 + F_0(T_0)}{1 + T_y} \sin y \quad (11.38)
\]

where the input modulation of the signal is assumed to be absent and \( F_0(y) \) is taken as

\[
F_0(y) = \frac{1 + F_0(T_0)}{1 + T_y} \quad \text{(11.39)}
\]

Referring to (10.35) one finds that the frequency acquisition time is given

\[
T_{AP} \approx T_y \left( \frac{\Omega}{F_0(0) AK} \right) \quad (11.40)
\]

and when the AFC circuit is cut off \( (K_y = 0) \), the corresponding time is given by

\[
T_F \approx \frac{T_F}{F_0(0) AK} \quad (11.40)
\]

The variations of \( T_{AP} \) and \( T_F \) with \( \Omega/2AK \) are shown in Fig. 11.4.

21.3 Injection Synchronized Phase Locked Loop

Consider the injection synchronized phase locked loop (ISPLL) as shown in Fig. 11.5. It consists of a mixer, a phase detector, two voltage-controlled oscillators VCO-1 and VCO-2, two low pass filters \( F_{dc}(s) \) and \( F_{ac}(s) \) and a \( \pi/2 \) phase shifter. The output of the mixer feeds both the \( \phi \)-detector and the oscillator circuit of the voltage-controlled oscillator VCO-2. The output of the VCO-2 passes to the other port of the phase detector through a \( \pi/2 \) phase shifter.
Fig. 11.4. The pull-in time characteristic of the modified PLL with the automatic frequency control arrangement.

Fig. 11.5. The PLL with an arrangement for injection synchronization.

The output of the phase detector is low-pass filtered to be fed to the input ports of VCO-1 and VCO-2. Note that the instantaneous frequency of the VCO-2 is controlled simultaneously by the error signal from the phase detector and the r.f. signal from the mixer output. Note further that the heterodyned output of the mixer feeds the phase detector. This arrangement effectively increases the linearity of the system, as the effective phase error at the input to the phase
detector becomes small. To derive the system equation let us consider:
that the input signal is accompanied by a narrow band noise, and
thus we write:
\[ n(t) = \sqrt{2} A \sin (\omega_d t + \phi_d) + n(t) \]  \hspace{1cm} (11.3)
where \( n(t) \) can be written as:
\[ n(t) = \sqrt{2} N(t) \cos (\omega_i t + \phi_i) - \sqrt{2} N(t) \sin (\omega_i t + \phi_i) \]  \hspace{1cm} (11.42)
and
\[ N(t) = n_i(t) \cos \phi_i + n_q(t) \sin \phi_i \]
\[ N_i(t) = n_i(t) \cos \phi_i - n_q(t) \sin \phi_i \]
Further assume that:
\[ \sigma = \sqrt{2} K_{in} \sin (\omega_d f + \phi_d) \]  \hspace{1cm} (11.45)
and
\[ \nu = \sqrt{2} K_{in} \sin (\omega_d f + \phi_d) \]  \hspace{1cm} (11.46)
Therefore, the outputs of the mixer and the phase detector are given by:
\[ n_1(t) = K_{in} \sin ((\omega_1 - \omega_d) t + \phi_1 - \phi_d) + K_{in} N_i(t) \cos ((\omega_1 - \omega_d) t + \phi_1 - \phi_d) \]  \hspace{1cm} (11.45)
and
\[ n_2(t) = (A \sin \phi + N_i(t) \sin \phi - N_q(t) \cos \phi) K_{in} F_1(t) \]  \hspace{1cm} (11.46)
where
\[ \sigma = \omega_1 - \omega_d - \omega_m(t) t + \phi_1 - \phi_d \]
Therefore,
\[ \frac{d\nu}{dt} = K_{in} F_1(t) \sin \phi + N(t) \]  \hspace{1cm} (11.47)
where
\[ N(t) = N_i(t) \sin \phi - N_q(t) \cos \phi \]
\( K \) denotes the sensitivities of both the VCO's.
Now referring back to the discussion on injection synchronization one finds that the mixer output, when injected to the VCO-2, modifies the instantaneous frequency by an amount \( \frac{\omega_d}{\omega_1} (\sin \phi + \phi(t)) \)
\[ K_{2}/K_{1} \text{. Therefore, the total frequency modulation of the VCO-2 is} \]
given by:
\[
\frac{df}{dt} = \frac{\omega_{d}K_{1}}{4QK_{1}} (\sin \phi + N(t)) + K_{2}K_{0}K_{2}F_{0}(\phi) (\sin \phi + N(t)) \tag{11.48}
\]

Thus combining (11.47), (11.48) and (11.49) one finds that
\[
\frac{df}{dt} = \Omega - \left( \frac{\omega_{d}K_{1}}{4QK_{1}} + \frac{KF_{2}(\phi) + \partial F_{2}(\phi)}{\partial \phi}(\sin \phi + N(t)) \frac{df}{dt} \right) \tag{1.50}
\]

where \( \Omega = \omega_{d} - \omega_{e1} - \omega_{e2} \) \tag{11.51}

Comparing (11.50) with the governing equation of a standard PLL \( 9.34 \) we find that it is similar to that of a conventional PLL where the signal from the phase detector is fed to the control element, i.e., the resistance modulator, through three elements, \( v_{c}, (1) \) with zero time constant, \( (2) \) with the filter \( F_{2}(\phi) \) and \( (3) \) with the filter \( F_{2}(\phi) \), but in practice it is almost impossible to realize a PLL having a controlling channel with zero time constant as the various elements of the PLL, such as the filtering circuit of the phase detector, i.e., de-encouraging circuit, the time constant of the de-amplifier when the PLL is used at microwave frequencies, etc., can act as a filter. This ultimately shortens the locking range of the PLL. Thus the circuit limitation on the capture capability of a PLL at higher band of frequency can be overcome with the provision of a direct synchronizing link.

To calculate the noise bandwidth of the system, when the output is observed at the output of the VCO-1, we assume \( \Omega = 0 \) and linearize the loop by putting \( \sin \phi = \phi \). Thus one finds from (11.50),
\[
\Phi(t) = \left( \frac{f(t)}{K_{0}K_{1} + KF_{1}(\phi) + KF_{2}(\phi)} \right) \frac{df}{dt} \tag{11.52}
\]

where
\[
E_{n} = \frac{\Phi}{Q^{2}K_{0}K_{1}}
\]

The noise phase modulation of VCO-1 is obtained from (11.48) and (11.53) as
\[ V_{in}(\delta) = \frac{K_F(\delta) - N(\delta)}{s + A(K_4 + K_F s(\delta) + K_F(\delta))} \] (11.53)

To simplify the calculation let us assume that

\[ F_1(\delta) = \frac{1}{1 + x F_1} \]

and

\[ s F_1 = \frac{1}{1 + x F_1} \] (11.54)

Therefore,

\[ V_{in}(\delta) = \frac{K_N(\delta)}{s F_1 + s(1 + A K_4 T_1 + A K_F F_1) + A K_4 + 2 A K} \] (11.55)

Therefore, the noise bandwidth is given by

\[ B_N = \frac{1}{4\pi} \int_{-\infty}^{\infty} H(\delta) |H(-\delta)| d\delta \] (11.56)

where

\[ H(\delta) = \frac{A K}{s F_1 + s(1 + A K_4 T_1 + A K_F F_1) + A K_4 + 2 A K} \] (11.57)

using (9.28b) one finds that

\[ B_N = \frac{AK}{(2 + AK_4 + AK)(1 + AK_4 T_1 + AK F_1 T_1)} \] (11.58)

which is much smaller than that of a simple PLL.

### 11.4 Test-time Modulated Signal Fed to the Modified Loops

In this section we will consider the performance of the extended range phase locked loop, frequency feedback phase locked loop and injection synchronized phase locked loop in an environment of high carrier to noise ratio. That is, we consider the situation when the loop is linearized.

#### 11.4.1 Extended Range Phase Locked Demodulator

Consider the extended range phase locked loop of Fig. 11.3 to be a first order one, so that (11.11) is written as...
\[ \Psi(t) = \frac{\Delta T}{s + AK(1 + sT_a)} N(t) \]  
(11.59)

The output of the phase detector is passed through a lowpass filter of the form \(1/(1 + sT)\). Thus the net output is given by

\[ \eta(t) = \frac{\Delta T}{1 + sT} + N(t) \]  
(11.60)

Thus comparing (11.60) and (11.59) one finds that

\[ \tau(t) = \frac{\Delta T}{1 + sT} \left( \frac{1}{1 + AKT_a} + AK \right) N(t) \]  
(11.61)

Referring to (9.136) through (9.141) and (11.61), it is easily seen that the signal to noise ratio at the output of a high carrier to noise ratio is the same as that of a simple PLL.

11.4.2 FREQUENCY FEEDBACK PHASE LOCKED DEMODULATOR

Consider the arrangement of FFPLD shown in Fig. 11.2. The output of the phase detector is passed through a lowpass filter of transfer function \(1/(1 + sT)\) to record the demodulator output. Thus, the output is written as (cf. 11.32)

\[ \eta(t) = \frac{\Delta T}{s + \Psi} G(t) + \Psi \]  
(11.62)

Take \(F(s) = 1\), and \(F(s) = 1\), and \(\eta(t)\) as

\[ \eta(t) = \frac{\Delta T}{1 + sT} G(1 + Kt) + K\delta \]  
(11.63)

Referring to (9.139) through (9.134) and (11.63) one finds that the (SNR) at the output of the FFPLD is similar to that of a conventional PLL at high CBR.

11.4.3 INJECTION SYNCHRONIZED PHASE LOCKED DEMODULATOR

Consider the situation when \(F(s) = 1\) and \(F(s) = 1\), and write the output as

\[ \tau(t) = \frac{\Delta T}{1 + sT} N(t) \]  
(11.64)

Compare this with (11.52) to find \(\eta(t)\) comes out as
\[ v_0(t) = \frac{A_P v(t) + sN(t)}{(1 + \alpha^2)(1 + AK_2 + AK_3 + AK_4)} \]  
(11-46)

Hence the conclusion is exactly as that for a conventional phase-locked loop.

11.5 Forced signal Acquisition

We have already seen that a first order loop quickly locks to an incoming signal without skipping a cycle, provided the frequency offset of the incoming signal does not exceed the locking range of the loop (i.e., AK). But a first order loop is almost never seen for its poor noise filtering properties. On the other hand, a second order loop is generally used for its better noise squelching property.

If one uses a second order type-2 loop, i.e., a loop with a perfect integrator, the limitation regarding the maximum frequency offset of the incoming signal does not arise. Because here the locking range is infinite. However, we have already seen that a second or third order type-2 loop requires a prohibitively large locking time if the frequency error is large. Again, for a second order type-1 loop the locking range is not only finite but the acquisition time is also very large when the frequency error is close to the locking range of the loop. Thus in either of the cases, it is desirable to use some aids for quickly locking onto the signal. There are various acquisition aids. We have already discussed some such methods in Section 11.2.3 by using a two mode control and in the following we will discuss another method [11,12].

11.5.1 Signal Acquisition by Sweeping the VCO Frequency

In order to improve the acquisition time, the swept frequency of the VCO is swept at a suitable rate. If the loop is a second order one, frequency sweeping can be done either by application of a sawtooth voltage at the VCO input (as shown in Fig. 11-6) or by application of a step voltage at the input to the filter. However, in the later case at the instant of the application of a step voltage, there will appear initially a voltage of approximately \(11P_2\) that the input step at the input to the VCO. This may throw the VCO frequency to a certain value at the beginning of the sweep. This may be undesirable for certain applications. As using, as the VCO is swept by the external agency, the pulling force on the oscillator
will be to the incoming signal as well as due to the sweep voltage. Consequently, frequency acquisition will be quicker. If the VCO frequency is swept by a ramp of slope of \( R \), then the output of the VCO may be assumed to be of the form \( 2\pi f_0 + R t \). Thus the loop equations for the second order type-2 and type-1 PLL are given by

\[
T_1 \frac{d\theta}{dt} + AKT_1 \cos \theta \frac{d\phi}{dt} + AK \sin \theta = -RT_1
\] (11.66)

and

\[
T_2 \frac{d\phi}{dt} + (1 + AKT_1 \cos \theta) \frac{d\phi}{dt} + AK \sin \theta = \Omega - RT_1 - RT
\] (11.67)

Therefore, for the second order type-2 PLL, the steady state phase error is given by

\[
\phi(2) = \sin^{-1} \left( \frac{\Omega}{\omega_0} \right)
\] (11.68)

whereas the phase error of the second order type-1 PLL, at the instant of completing acquisition, is

\[
\phi(1) = \sin^{-1} \left( \frac{\Omega}{AK} - \frac{R}{\omega_0^2} - \frac{R_1}{AK} \right)
\] (11.69)
Looking at (11.66), it appears that the presence of the sweep voltage after acquisition has been achieved, does not throw the system out of lock. However, the presence of the sweep voltage will ultimately cause saturation of the DC amplifiers in the loop. Moreover, the presence of the sweep voltage after acquisition will cause difficulty in re-acquisition if the signal fades out for a while. Because, during the fading interval the VCO frequency will be carried off to some distance and it is likely that the VCO frequency will be swept further. This calls for removal of the sweep voltage as soon as the acquisition is complete. Referring to (11.69) one finds that if the sweep voltage is not removed after the acquisition, the loop will be thrown out of lock. The removal of the sweep voltage is achieved by means a decision circuit which incorporates a lock-in indication system along with a switching device.

Sweep acquisition behavior of a second order type-2 PLL has been investigated by Yiterbi [8, 13]. Although the relation (11.69) indicates that locking should be possible if $\Delta \omega > 0$, Yiterbi has shown that in this case locking probability in unity only when $R$ is somewhat less than 0.5 $\Delta \omega$. This happens because the chance of locking and non-locking depends on the random initial conditions of the phase and frequency. Probability of lock reduces expeditiously to zero as $R$ approaches $\Delta \omega$. It might appear that the damping of the loop does not have any effect. Frazier and Page [9] has shown that increase in the loop damping factor increases the probability of acquiring lock.

Now in the real world, the signal is always accompanied by noise. In such a situation, the probability of acquiring lock is reduced, and as such the relation between the slope of the sweep voltage $R$ and the loop natural frequency $\omega_0$ will be different from that of the no noise case. Frazier and Page have obtained an empirical relation between the maximum sweep rate and the loop natural frequency that will ensure an acquisition probability of 0.9. This relation is

$$R(\text{max}) = \omega_0^2 \left(1 - \frac{1}{\pi p^2}\right) \frac{1}{1 + \exp \left(-\frac{\pi^2 p^2}{4} \left(1 - \frac{\pi^2 p^2}{4}\right)\right)} \text{ for } \xi < 1$$

(11.70)

and

$$R(\text{max}) = \omega_0^2 \left(1 - \frac{1}{\pi p^2}\right) \text{ for } \xi \geq 1$$

(11.71)

where $p$ is the signal to noise ratio in the loop bandwidth (cf. 9.72). The above relation does not at all agree with the theoretical results
of Viterbi for noise-free situation. For example, according to (11.70), a loop with a damping factor of 0.707 will require a maximum sweep rate of 0.588 for a very high value of $e$, whereas Viterbi’s result is nearly 0.7 of Frazier and Page also considered the situation when a PLL incorporates a bandpass limiter in the IF stage. They found an empirical solution of the form

$$R_{\text{max}} = \frac{1.24 \left( \frac{e_0}{\nu_0} \right)}{1 - 9/\sqrt{2} \left( 1 + \exp \left( -\frac{\nu_0}{\nu} \right) \right)} \text{for } \xi \ll 1$$

(11.72)

where $e$ is the signal suppression factor due to the bandpass limiter (cf. Chapter 15). $N_0$ is the signal suppression factor measured at threshold SNR, and $\nu_0$ is the loop natural frequency at the threshold SNR. For further results on this topic, the reader may refer to the works of Vaila [10] and Salanchak [11].

Nothing is known regarding sweep acquisition analysis for signals coming through turbulent media such as the turbulent interplanetary plasma, the ion plasma of an electron propulsion engine, etc., that exhibit Rayleigh, Rician and nongaussian fading, or signals from tumbling satellites that exhibit periodic and frequent fading.

REFERENCES

CHAPTER 12
HETEROODYNE AND MULTIFILTER LOOPS

In practice the standard or conventional circuit configurations using a phase detector, a low pass filter and a voltage controlled oscillator, is often modified in order to improve the noise handling capability of a phase locked receiver. For example, a phase locked receiver often employs heterodyning of the incoming signals in order to utilize the advantages of intermediate frequency amplification. Again in Appolo mission, where the modulating signal consists of a number of subcarriers in addition to a wideband video, bandpass design of a PLL is recommended to realize improved performance. These changes in the basic loop configuration can modify the loop behavior to a great extent and in this chapter we will discuss the properties of such loops.

12.1 The Heterodyne Phase Locked Loop

A typical configuration of a heterodyne PLL is shown in Fig. 12.1.

![Diagram of a heterodyne PLL](image)

Fig. 12.1 A typical PLL with the bandpass filter in the loop.

The output of the VCO, $\sqrt{2}\xi \cos (\omega t + \psi(t))$, after frequency multiplication by a factor of $N$, is mixed with the incoming signal $\sqrt{2}\xi \sin (\omega t + \psi(t))$. The difference frequency output of the mixer
is phase detected with the help of a crystal sensitive detector, having an output of the form \( v_1(t) = \omega_0 t + \Psi(t) \). Although the IF amplifier stage is a conventional one, its component units are shown separately for the sake of clarity.

The difference frequency output of the mixer is \( \Delta f_\text{b} \sin (\omega_0 t - N\omega_0) \).

Therefore, the output of the IF amplifier is given by

\[
v_1(t) = A\Delta f_\text{b} \left( \sin (\omega_0 t - N\omega_0) \sin \Psi(t) \right)
\]

where \( f(t) \) is the impulse response of the IF amplifier. \( v_1(t) \) may be written as

\[
v_1(t) = A\Delta f_\text{b} v(t) \sin (\omega_0 t - N\omega_0) + \Psi(t)
\]

where \( (A\Delta f_\text{b} v(t)) \) and \( \Psi(t) \) are respectively amplitude and phase functions of \( v_1(t) \), and \( \omega = \omega_0 - N\omega_0 \).

Therefore, the output of the phase-detector can be written as

\[
v_p = A\Delta f_\text{b} [\Psi(t) + \Psi(t)]
\]

where

\[
v_p(t) = \left( \omega_0 - \omega - N\omega_0 \right) t + \Psi(t)
\]

The output of the low pass filter can be expressed as

\[
u_\text{d}(t) = \int v_p(t) f(t - \omega) d\omega = P(f) u_\text{d}(f)
\]

where \( f(t) \) is the impulse response of the low pass filter.

Thus

\[
\frac{d\Psi}{dt} = A\Delta f_\text{b} K_\text{d} K_\text{p} \int P(f) f(t) \sin \Psi(f) df
\]

Note this is a short-hand way of writing the equation, using Havriliak's operator \( p = \frac{d}{dt} \).

Next putting

\[
\Psi = (\omega_0 - \omega - N\omega_0) t + \Psi(t) - N\Psi(f)
\]

and comparing (12.6) and (12.7) one finds that
\[ \frac{d\phi}{dt} = -\Omega - AK\cdot NP\phi(t) \sin q_0(t) + \frac{dV_t}{dt} \] (12.8)

where

\[ \Omega = \omega_0 - \omega_1 - N\omega \]

and the other symbols have their usual significance.

From (12.8) it is clear that the property of a heterodyne loop can be analyzed only when \( \phi(t) \) and the relation between \( q(t) \) and \( q_0(t) \) are known. If the bandwidth of the IF amplifier, having unity gain, were large, then \( \phi(t) \) could be equated to unity and \( q_0(t) \) to \( q(t) \). In general, the characterization of the impulse response of a practical IF amplifier is difficult, and as such we will consider the following cases.

**CASE I: Very large IF amplifier bandwidth**

Obviously, in this case \( \phi(t) = 1 \) and \( q_0(t) = q(t) \). Thus (12.8) could be written as

\[ \frac{d\phi}{dt} = -AKNP\sin q + \frac{dV_t}{dt} \] (12.9)

The phase governing equation (12.9) is similar to that of a conventional phase-locked loop in which the loop gain has increased by the factor of the frequency multiplication factor \( N \). Aside from this, the loop has two advantages, viz., (i) phase detector can be operated at a much lower (negligible) than the incoming signal thus avoiding circuit complexity at h.f. and (ii) after acquisition is achieved (\( \Omega = 0 \)), one notes that \( \omega_0 - N\omega = \omega_0 \). This indicates the centre frequency of the mixer output becomes equal to that of the reference signal derived from a crystal oscillator. Therefore, the centre frequency of the IF signal becomes extremely stable. This implies that the bandwidth of the IF amplifier can be considerably narrowed down to improve the signal-to-noise ratio in the loop. But narrowing down the IF bandwidth beyond a certain limit creates complication in the loop operation leading to misaligning behaviour, known as false locking. We will consider this in the sections to follow.

**CASE II: Single tuned IF amplifier**

Let us assume that the heterodyne loop incorporates a single tuned IF amplifier. A single tuned IF amplifier of unity gain and the quality factor \( Q \), has the following transfer function
Heterodyne and Multiplier Loops

\[ H(\omega) = \frac{1}{1 + j2Q\frac{\omega - \omega_0}{\omega}} \]  

(12.10)

where \( \omega_0 \) is the centre frequency of the IF amplifier. If the \( Q \) factor is high compared to unity, we may approximate (12.10) as

\[ H(\omega) = \frac{1}{1 + j2Q(\omega - \omega_0)\omega > 0} \]  

(12.11a)

and

\[ H(\omega) = \frac{1}{1 + j2Q(\omega + \omega_0)\omega < 0} \]  

(12.11b)

Thus the lowpass or baseband equivalent transfer function of the IF amplifier is written as

\[ F(\tau) = \frac{1}{1 + \pi T} \]  

(12.12)

where

\[ T = \frac{2Q}{\omega_0} \]  

(12.13)

\( \pi(t) \) sin \( \phi(t) \) can be written as

\[ \pi(t) \sin \phi(t) = \frac{AKN}{1 + pT} \sin \eta(t) \]  

(12.14)

Therefore, the loop equation (12.8) may be written as

\[ \frac{d\phi}{dt} = \Omega - \frac{AKN}{1 + pT} F(\tau) \sin \eta + \frac{dV_x}{dt} \]  

(12.15)

Looking at (12.12) one can conclude that the behaviour of the loop will be modified because of the presence of \( \frac{1}{1 + \pi T} \). It definitely changes the stability criteria as well as the locking characteristics of the loop.

We have seen in Chapter 8 that a conventional Ph.L, incorporating a proportional plus integrating filter or an integrating filter (type-2 and type-1 loops) is unconditionally stable. Let us now look into the stability of a heterodyne loop incorporating the above type of filter. The transfer function of the heterodyne loop can be written as
\[ H_0(\lambda) = \frac{AKNF(\lambda)(1 + \lambda T)}{\lambda + AKNF(\lambda)(1 + \lambda T)} = \frac{AKNF(\lambda)}{\lambda + AKNF(\lambda)} \]  \tag{12.16}

For a proportional plus integrating filter network \( u(\tau) = \frac{1 + \lambda T}{\lambda + \lambda T} \), the relation (12.13) can be written as

\[ H_0(\lambda) = \frac{AKNF(\lambda)}{\lambda + AKNF(\lambda)} \]

i.e.,

\[ G(\lambda) = \frac{AKNF(\lambda)}{\lambda + AKNF(\lambda)} \]

\[ = \frac{AKNF(\lambda)}{\lambda + AKNF(\lambda)} \tag{12.17} \]

Note that the form of the denominator polynomials of (12.17) is similar to that of Case II of Section 8.2. This indicates that the presence of the IF filter makes the loop conditionally stable. The root locus plot is exactly similar to that of Fig. 8.1d. Following the procedures of Section 8.2, it can be shown that for stability the following relation it to be satisfied

\[ AKN < \frac{T + T_i}{T + T_i - T_i \lambda - T_i \lambda^2} \]

\[ = \frac{T_i + T}{T_i} \tag{12.18} \]

where

\[ G(p) = F(p) \frac{1 + \lambda T}{\lambda + \lambda T} \tag{12.19} \]

We assume that the loop is operating under unlooked condition, and approximate [1] the solution (1, 2, 3) of (12.19) as

\[ \phi = \beta u(t) \frac{1 - \lambda T}{1 + \lambda T} \sin \omega t \tag{12.20} \]

Substituting this value of \( \phi \) in (12.19) using the method of harmonic balance, one finds (see Section 10.4)

\[ \Omega = \omega - AKNF(\lambda) \sin \omega \]

\[ \lambda \omega = AKNF(\lambda) J_1(\omega) \sin (\omega - \phi(\omega)) + J_1(\omega) \sin (\omega + \phi(\omega)) \]
where \( G(\omega) = G(\omega) \exp(-j\omega) \)  

\( (12.23) \)

Since the value of \( \omega \) is small in the beating condition, we neglect \( j\omega \) in comparison to \( \omega(\omega) \) and put \( j\omega = n2 \) to the above equations. Thus we get

\[
\omega = \pm \sqrt{2 + \phi(\omega) \frac{AEN(\omega)}{\omega}} \tag{12.23}
\]

and

\[
\Omega = \omega + \frac{AEN(\omega)}{2\omega} \sin \phi(\omega) \tag{12.24}
\]

which can be written as \( (\omega = \omega(AEN) \omega(\omega)) \)

\[
\frac{\Omega}{AEN} = \omega + \frac{1}{2\omega} \sin \phi(\omega) \tag{12.25}
\]

For the particular case when

\[
F(\omega) = \frac{1}{1 + f_N \sin^2 \angle \frac{\omega}{\omega}}
\]

one finds that

\[
G(\omega) = \frac{1 + \omega(\omega)^2}{1 + \omega(\omega)^2} \quad \frac{\omega}{\omega} + \frac{1}{1 + \omega(\omega)^2} \tag{12.27}
\]

\[
\phi(\omega) = -\arctan \left( \frac{\omega(\omega)^2}{\omega(\omega)^2 + \omega(\omega)^2} \right)
\]

\[
\phi(\omega) = -\arctan \left( \frac{\omega(\omega)^2}{\omega(\omega)^2 + \omega(\omega)^2} \right) \tag{12.28}
\]

Locking range of the loop in such a case can be found from (12.27) or (12.28) by finding the value of \( \omega \) for which there is no solution for the beat angular frequency \( \omega \). Referring to Fig. 12.2, it is readily appreciated that the following steps be taken to find the locking range: (i) first, find the value of \( \omega \) at which the right hand side of (12.26) becomes minimum, and (ii) then substitute this value of \( \omega \) in the transition beat angular frequency in (12.27) to evaluate the locking range. To illustrate this, let us consider the situation when the effect of IF filter is ignored, and suppose \( F_N \) is small and \( T_N \) is large. In this case (12.25) can be approximated as

\[
\Omega = \omega + \frac{\omega(\omega)^2}{2\omega} \tag{12.29}
\]

The variation of \( \omega \) with \( \Omega \) for this case is shown by the curve (3) of
Fig. 12.2. Illustrating the method of base-frequency calculation for a PLL with delay elements.

Fig. 12.2. Differentiating this with respect to \( \omega \) and equating to zero, one finds that the transitional value of \( \omega \) is given by

\[
\omega_0 = \frac{AK\sqrt{B}}{2}
\]

(12.30)

Therefore, the locking range \( (\Omega_m) \) is obtained from (12.29) and (12.30) as

\[
\Omega_m = AK\sqrt{2E_0}
\]

(12.31)

This agrees with the earlier results on the value of the locking calculated in Chapter 10.

Referring to the curve (1) Fig. 12.2, it is seen that for a value of \( \Omega \) outside the locking range, there are two values of \( \omega \). The value \( \omega_1 \) is unstable, as it indicates that a decrease in \( \Omega \) increases the beat angular frequency \( \omega \). Whereas the value \( \omega_0 \) with positive slope, is stable. Again referring to Fig. 12.2, one finds that an increase in the value of \( T \), i.e., narrowing down the IF bandwidth, causes the minimum value \( \Omega_m \) (locking range) to approach zero and finally to cross the \( \omega \)-axis. The value of \( T \), that makes the locking range to vanish (curve-2), may be found from (12.25) and (12.26).

First, differentiate (12.25) with respect to \( \omega \) and put it equal to zero i.e.,
1 - \frac{\mathcal{A}KNP}{\omega_0} \left( \frac{G(\omega_0)}{\omega_0} \cos \theta(\omega_0) + G'(\omega_0) \cos \theta(\omega_0) \right) + \frac{G(\omega_0)}{\omega_0} \sin \theta(\omega_0) = 0 \tag{12.22}

where \( \theta(\cdot) \) denotes differentiation with respect to \( \omega \).

Secondly, at \( \omega_0 \), the left hand side of (12.22) is zero, i.e.,

\[ \omega_0 + \frac{\mathcal{A}KNP}{\omega_0} \frac{G(\omega_0)}{\omega_0} \cos \theta(\omega_0) = 0 \tag{12.23} \]

Eliminating \( \omega_0 \) from (12.22) and (12.23) one can find the critical value of \( T_0 \) (say, \( T_{cr} \)), for which the locking range vanishes.

Now let us study the nature of variation of the dc voltage at the output of the phase detector. Referring to the relations (12.4) and (12.18), one finds that

\[ \sin \theta = \sin(\omega t + \alpha) \cos(m \sin \omega t) + \cos(\omega t + \alpha) \sin(m \sin \omega t). \]

Therefore, using the relations

\[ \cos(m \sin \omega t) = 2 \sum_{k=0}^{m} j^k \cos(2k+1) \omega t \]

and

\[ \sin(m \sin \omega t) = 2 \sum_{k=0}^{m} j^k \sin(2k+1) \omega t \]

it can be shown that the dc component of \( v_\phi \) is given by

\[ v_\phi |_{\omega t} = A_\Delta K_i \left( \frac{\mathcal{A}KNP}{\omega} \right) \cos \theta(\omega) \tag{12.34} \]

Further from (12.19) to (12.24) one has

\[ \Omega = \omega_{0} + \frac{\mathcal{A}KNP}{\omega} \cos \theta(\omega) \tag{12.25} \]

Variation of \( v_{d,0} \) with the open loop frequency error is shown in Fig. 12.3. From the Fig. 12.3 one finds that as the time constant of the low-pass equivalent of the IF filter increases beyond a critical value the average dc potential at the output of a phase detector becomes negative for a positive open loop frequency error. This means that the pulling effect, which causes synchronization, will be absent, and instead there will be a pushing effect, so to say, and synchronization will fail.

The effect of the single tuned IF filter on the noise Skimming pr-
calculate the noise bandwidth, let us look into the system equation in the presence of noise. Following the method of Chapter 9, it is easily shown that the mixer output is now given by

$$v_n(t) = K_1 \int \frac{[A \sin \theta(\omega) + N(\omega)] f_0(t - \omega) \, d\omega}{\delta} \tag{12.36}$$

where

$$0 = (\omega - \Omega_N) t + \varphi_N - \varphi \tag{12.37}$$

$$N(t) = N_0 \cos \theta - N_0 \sin \theta \tag{12.38}$$

and $$f_0(t)$$ is the impulse response of the IF filter.

Following the procedure of Chapter 9 and utilizing the relations (12.11) to (12.12) it can be easily shown that the system equation in the presence of the incoming noise is given by

$$\frac{d\varphi}{dt} = \Omega + AKN \frac{F(p)}{1 + p^2} \left[ \sin \varphi + N(p)/A \right] + \frac{d\varphi}{dt} \tag{12.39}$$

Therefore, the noise bandwidth of the PLL, incorporating the IF filter, is given by

$$B = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| H_0(\omega) \right|^2 \, d\omega \tag{12.40}$$

Fig. 12.3. Variation of the average $\phi$ detector output with the open loop frequency error for a PLL with noise.
where $H_0(f)$ is given by (12.13). For a PLL with a proportional plus
integrating filter, one finds from (12.16) and (12.20) that

$$4B\{AKN + \frac{T}{T + T_1 + AKNP}T I\} = T + T_1 + AKNP + T I - AKNP(1 - \frac{T_1}{T})$$

(12.41)
i.e.,

$$4B\{AKN = T + T_1 + AKNP T I\}$$

(12.42)
For stable operation of the PLL, it is necessary (cf. 12.18) that

$$T + T_1 > AKN(T + T_1 + 3 - T_0 + 3)$$

(12.43)
If one considers a PLL with an integrating filter of the form \(1 + 2T_{\delta} / T\),
the noise bandwidth of such a loop is easily shown to be (cf. 12.16) and (12.40)

$$B = \frac{\delta^2}{4} \left( \frac{1}{T_{\delta}} + 1 \right)$$

(12.44)
Obviously for stability, it is necessary that $2T_{\delta} > \omega_0 T$. This is easily
proven by applying the criteria of stability of a third order loop.

For a fixed value of the delay time $T$, the noise bandwidth (12.43)
can be minimized by varying the value of $\delta$. The minimum value of $B$ and the corresponding value of the damping factor $\delta$, can be
found by putting

$$\frac{\partial B}{\partial \delta} = 0$$

whence one finds that

$$2T_{\delta} = \omega_0 T + \sqrt{1 + \omega_0^2 T^2}$$

(12.45)
giving the minimum value of $B$, which is seen to be

$$B_{\text{min}} = \frac{\omega_0 T}{3 - \sqrt{1 + \omega_0^2 T^2}}$$

(12.46)
Had there been no delay element in the loop, the minimum noise
bandwidth would have been $\omega_0 T$. Thus it appears that the noise
bandwidth increases with increasing value of $T$. The variation of $B_{\text{min}}$ with $\omega_0 T$ is shown in Fig. 12.4.
320 Phase Lock Theories and Applications

Fig. 12.6. Variation of the minimum noise bandwidth with the equivalent time constant of the IF filter.

**CASE III:** Use of Crystal Filter

In many cases, the mixer circuit uses a crystal bandpass filter at the IF stage. These crystal filters may be assumed to have a uniform gain and linear phase shift 0 over the frequency band of interest. Then the IF filter may be assumed to produce a uniform gain $K_f$ (say, unity) and a time delay $T = \frac{\lambda_0}{2f_0}$ [5]. In this case, it is easily shown that the final phase equation (cf. 12.6) is given by

$$\frac{d\phi(t)}{dt} - \Omega - AKNF(p) \sin \phi(t - T) = \frac{d\phi_s}{dt}$$  \hspace{1cm} (12.47)

In this subsection we will consider the locking characteristics and as such we will ignore $d\phi_s/dt$. That is, we rewrite (12.47) as

$$\frac{d\phi(t)}{dt} = \Omega - AKNF(p) \sin \phi(t - T)$$  \hspace{1cm} (12.47a)

Consider first the case when no additional filter network is used, i.e., $F(p) = 1$, and we get

$$\frac{d\phi(t)}{dt} = \Omega - AKN \sin \phi(t - T)$$  \hspace{1cm} (12.47b)

If the delay were absent ($T = 0$), we have already seen that the system will lock onto the signal provided $\Omega < AKN$. Otherwise it will
nor. Equation (12.47b) is a nonlinear difference-differential or hysteresis-differential equation of first order or equivalent to nonlinear differential equation of infinite order. No analytical solution of such an equation is available. However, graphical [6, 7] or numerical solution via a digital computer can be had.

As usual let us begin with the assumption that the phase error $\epsilon$ is small so that we can replace $\sin x$ by $x$ without much error. Thus (12.47b) is reduced to

$$\frac{d\phi(t)}{dt} + A\phi(t - T) = \Omega$$

This equation can be solved [8, 9] with the initial conditions

$$\phi(t - T) = \phi_0, \quad 0 \leq t \leq T$$

Thus

$$\frac{d\phi(t)}{dt} = (\Omega - A\phi_0), \quad T \leq t \leq 2T$$

i.e.,

$$\phi(t) = (\Omega - A\phi_0) t + \text{constant}$$

Since at $t = T, \eta(0) = \phi_0$, one gets

$$\eta(t) = \phi_0 + (\Omega - A\phi_0)(t - T), \quad T \leq t \leq 2T$$

(12.50)

Thus using (12.48) and (12.50) one writes

$$\frac{d\eta(t)}{dt} = (\Omega - A\phi_0) - A\phi(t - T), \quad \text{for } T \leq t \leq 3T$$

(12.51)

Integrating this and using the boundary conditions at $t = 2T$ (cf. 12.50)

$$\eta(2T) = \phi_0 + (\Omega - A\phi_0) T$$

one finds that

$$\eta(t) = \phi_0 + (\Omega - A\phi_0)(t - T) \quad \text{for } 2T \leq t \leq 3T$$

(12.52)

Therefore, from (12.52) and (12.48) one gets

$$\frac{d\eta(t)}{dt} = \Omega - A\phi(t - T), \quad \text{for } 3T \leq t \leq 4T$$

(12.53)
Integrating (12.53) and using the initial condition at $t = 3T$,

$$\theta(3T) = \theta_0 + \Omega - AK\theta_0)^3T - \frac{A\Omega - AK\theta_0}{2} 4T$$

one finds that

$$\theta(t) = \theta_0 + (\Omega - AK\theta_0)^3T - AK\theta_0^3T (t - 2T)^3$$

$$+ \frac{AK^3}{12}(1 - AK^2(t - 2T)^3)$$

$$= \frac{\Omega}{AK} - \left(\frac{\Omega}{AK} - \theta_0\right)\left[1 - AK^2(t - 2T)^3\right]$$

$$- \frac{1}{12} AK^3(1 - 3T)$$

(12.54)

Proceeding in this way one can generalize the solution in the following form

$$\theta(t) = \frac{\Omega}{AK} - \left(\frac{\Omega}{AK} - \theta_0\right)\frac{(t - T)^3}{3!} (AK)^3$$

$$+ \frac{AK^3}{12}(1 - AK^2(t - 2T)^3)$$

when the delay $T$ is ignored, we know that the solution is

$$\theta(t) = \frac{\Omega}{AK} - f\left(\frac{\Omega}{AK} - \theta_0\right) \exp(-AKt)$$

(12.27)

The variation of $\theta(t)$ with time for the two cases are shown in Fig. 12.5.

12.2 Nonlinear Behaviour

In this section we discuss the non-linear behaviour [9, 7] of a first order delayed phase-locked loop with a sinusoidal phase detector, for which the system equation is given by

$$\frac{d\theta(t)}{dt} = \Omega - AK \sin(\theta(t) - T)$$

(12.55)

For the sake of convenience, $\theta$ has been abstained in $AK$. When the delay is absent, we know that the phase plane plots can be either of the types for a positive damping as shown in Fig. 12.6.

This depends on whether $\Omega$ is either less or greater than $AK$. When $\Omega < AK$, the loop is locked around the stable equilibrium point $A$. 
With these few words about the operation of a delayed PLL, let us turn to the case of a delayed PLL. Since the differential equation is not amenable to analytical solution, it is solved by a digital computer and the phase plane plots are shown in Fig. 12.7 for various values of $\Omega/\Delta K$. In these diagrams, the phase plane plot for the case when $\Omega > \Delta K$ is not shown, because the system is oscillating outside the locking region. We will consider this at a later stage. Fig. 12.7 shows the cases when $\Omega/\Delta K$ is gradually reduced from unity.
Case I: \( \frac{\Omega}{AK} < \frac{\Omega}{AK} < 1 \)

Since in this case \( \frac{\Omega}{AK} \) is less than unity it is expected that the PLL will be synchronized. However, due to the presence of the delay time \( T \), the loop will not be pulled into synchronization till \( \frac{\Omega}{AK} \) becomes less than certain critical value \( \frac{\Omega}{AK} \), the value of which depends on the delay time. The trajectory in this case is shown by the solid line of Fig. 12.7b. This is a periodic curve and is called the limit cycle of the second kind [6].

Fig. 12.7 (a)

Case II: \( 0 < \frac{\Omega}{AK} < \frac{\Omega}{AK} \)

In this case the loop is synchronized. The phase plane trajectories in this case are shown in Fig. 12.7b and Fig. 12.7c. In the case of Fig. 12.7b, the operating point tends towards a stable point A. Whereas in the case of Fig. 12.7c, the trajectory finally moves along another closed curve \( C \). Obviously this represents instability around the point A. Whether this limit cycle, called the limit cycle of the first kind, will appear or not, depends on the value of the delay time \( T \).

Fig. 12.7 (b)
Fig. 12.7. The phase plane plots of a first order PLL with time delay.

Case III: \( \Omega/\Delta K = (\Omega/\Delta K)_c \).

In this case, if the trajectory starts at \( A \), it will automatically end at the next point \( A \). In the above cases, we have observed the various phenomena by varying \( \Omega/\Delta K \) and keeping \( T \) to remain at an optimum value. However, the above operation may be repeated by varying \( T \), instead of changing \( \Omega/\Delta K \).

From what has been said it is clear that the PLL does not fall in synchronization with the external signal unless the value of \( \Omega/\Delta K \) equals its critical value \( (\Omega/\Delta K)_c \). This indicates that there appears a 'pull-in' phenomenon. Similarly, when the value of \( (\Omega/\Delta K)_c \) becomes greater than its critical value, the system pulls out of synchronization.

12.3 Stability

In order to consider the stability of the system at the singular points \( A \) or \( B \), we assume that initially the system is at one of those points. That is,

\[ \sin \phi = \frac{\Omega}{\Delta K} \]  \( (12.59) \)
We then increase the value of \( \theta_0 \) to \( \varphi_0 + x \), where \( x \) is an arbitrarily small quantity. Thus putting

\[ \varphi = \varphi_0 + x \]

in (12.57) one gets after using (12.58)

\[
\frac{dq}{dt} + AK \cos \eta, m(t - T) = 0
\]

(12.60)

solution of (12.60) can be easily written as (cf. 12.58)

\[ x(t) = x_0 \sum_{n=0}^{\infty} \left( \frac{T}{\pi + 1/2} \right)^n \left( -AK \cos \eta \right)^n N T < t < (N + 1/2)T \]

(12.61)

where \( x_0 \) is an initial value of \( x \). The point \( B \) is unstable point since \( \cos \eta \) is negative at this point.

To consider more on the point of stability, let us solve (12.60), which is considered to be a linear differential equation of infinite order, and thus its solution can be assumed [6] to be of the form \( x_t \exp(\sigma t) \). Substituting this in (12.60) one gets

\[ p = -AK \cos \eta, \exp(-\rho T) \]

(12.62)

which may have real and complex roots. Before going into the details of solving (12.63), we rewrite it as

\[ ze^{-\theta} - q = 0 \]

(12.62a)

where

\[ z = \eta T \]

and

\[ q = -AK \cos \eta \]

Now we quote a result of Hayes [3] which says that

"All the roots of \( az^2 + q - ze^\theta = 0 \), where \( p \) and \( q \) are real, have negative real parts if and only if

(a) \( a < 1 \), and

(b) \( a - \theta < \sqrt{a + p} \)

where \( \theta \) is a root of \( b = \arctan(b) \) such that \( \theta < b < \pi \), if \( a = 0 \), we take \( b = \pi/2 \)."

Thus we find from (12.62a) that the point \( A \) could be a stable point provided

\[ 0 < \rho T \cos \eta \leq \pi/2 \]

(12.63)

Now putting
\[ z = \sqrt{T} = \xi + j\eta \quad \text{and} \quad AKT \cos \eta = M \] (12.64)

One gets

\[ \xi + j\eta = -M \exp(-\xi - j\eta) \] (12.65)

Equating real and imaginary parts, one obtains

\[ \xi = -M \exp(-\xi) \cos \eta \] (12.66)

\[ \eta = -M \exp(-\xi) \sin \eta \] (12.67)

From which one gets

\[ \xi^2 = M^2 \exp(-2\xi) - \xi^4 \] (12.68)

\[ \xi = -\eta \cot \eta \] (12.69)

These equations can be graphically solved and \( \psi \) is illustrated in Fig. 12.8. Since the curves are symmetrical with respect to the axes \( \eta = 0 \),

**Fig. 12.8.** Illustrating the evaluation of the roots of the characteristic equation: \( p + AK \cos \psi \exp(-pT) = 0 \).
one half of the plane is shown. The intersection of the two sets of curves gives the roots of the characteristic equation. Depending on the location of the roots, stability of the solution is easily inferred. Solutions, having negative value of $\zeta$, give stable modes. This is illustrated in Fig. 12.9.

### 12.4 False Acquisition

In this section we discuss the behavior of a PLL incorporating a narrowband IF filter, e.g., a crystal filter. In this case, the operation of the loop during acquisition is considerably modified. As a result, the loop is sometimes locked to frequencies other than the correct frequencies causing what is known as false or spurious locking (5-8). The analysis presented here follows closely to those of the earlier workers (5, 9).

Let us consider a PLL with a sinusoidal phase detector and the loop incorporating a proportional plus integrating filter with a large time constant and asymptotic gain $F_0$. Let us further assume that the loop is operating in the unlocked mode. Under this condition, we have seen that the output of the phase detector consists of a dc voltage and alternating voltage component. As a result, the loop filter, it may be thought that the output of the filter consists of a dc component and only the fundamental component of the ac voltage. Thus we write the output of the VCO as

$$\eta = \sqrt{2}A_0 \cos [(N_0 + U)t - M \cos (G_0 - V)]$$

(12.70)

Here $U$ signifies the frequency shift of the VCO the assumed ac component at the phase detector output, whereas $M \cos (G_0 - V)$ appears because of the ac voltage. Note that the input to the PLL has been assumed to be of the form

$$\eta = \sqrt{2}A \sin \omega t$$

Obviously, the modulating frequency $\omega_1$ is given by

$$\omega_1 = \omega_0 - N_0 - U - o_2$$

(12.71)

It is important here to note that the phase angle $V$ is introduced as a result of the phase shift of the signal through the IF filter and other loop components, the latter being small compared to the former. Now the output of the phase detector can be written as
\[ v_p = AK_0 K_1 \sin \Omega t + M \cos (\Omega t - \psi) \]  
(17.72)

\[ = AK_0 K_1 \sin \Omega t \cos (M \cos (\Omega t - \psi)) + AK_0 K_1 \cos \Omega t \sin (M \cos (\Omega t - \psi)) \]

i.e., because of the low pass filtering arrangement

\[ v_p = (\mathcal{F}(M) AK_0 K_1 \sin \Omega t) + (AK_0 K_1 \mathcal{F}(M) \cos \psi) \]  
(17.72a)

The \(2\Omega_1\) and higher frequency terms have been ignored.

Since the output of the filter modulates the instantaneous frequency of the VCO, one finds from (12.70) and (12.72a) that

\[ M\Omega_1 = AK_0 K_0 K_1 \mathcal{F}(M) F_0 \]  
(12.73)

because the output of the filter for the \(\alpha\) component is \(AK_0 K_1 \mathcal{F}(M) F_0\). Similarily, we write

\[ U = AK_0 K_0 K_1 \mathcal{F}(M) \cos \psi \]  
(12.74)

since \(U\) is due to the dc voltage at the output of the phase detector.

Since \(F_0\) is small and the loop operates outside the lock-range \(\Omega > AK_0 K_0 K_1\), one finds that (cf. 12.73)

\[ M = \frac{4AK_0}{\Omega_1} \]  
(12.75)

Therefore, the dc output of the phase detector, which is responsible for pull-in phenomenom in a PLL, is given by

\[ v_p = AK_0 K_1 \frac{4AK_0}{\Omega_1} \cos \psi \]  
(12.76)

Further comparing (12.74) and (12.71) one finds that

\[ \Omega_1 = 0 - AK_0 \frac{AK_0}{\Omega_1} \cos \psi \]  
(12.77)

For all practical purposes, \(J_1 \frac{AK_0}{\Omega_1}\) may be replaced by \(J_1 \frac{AK_0}{\Omega_1}\), thus from (12.77) one finds that

\[ 2\Delta \Omega = 2\Delta \Omega_1 + AK_0 K_1 \cos \psi = 0 \]

i.e.,

\[ \Omega_1 = \Omega \pm \sqrt{\Omega^2 - \frac{2AK_0 K_1 \cos \psi}{\Delta \Omega}} \]  
(12.78)

In deriving this relation it has been assumed that \(\psi\) depends on \(\Omega\) (although it depends on \(\Omega_1\)). This is fairly correct for low values of \(\psi\).
By virtue of the assumption $\Delta K \Omega_0 \ll 1$ one gets

$$\dot{\Omega}_2 = \Omega - \frac{\Delta K \Omega_0^2}{2\Delta} \cos \Psi$$

i.e.,

$$\frac{\dot{\Omega}_1}{\Delta K} = \frac{\Omega}{\Delta K} - \frac{\Delta K}{2\Delta} \cos \Psi \tag{12.79}$$

Now when $\Psi$ is constant, i.e., there is no delay in the IF stage, the variation $V_y$ with $\Omega/\Delta K$ is shown in Fig. 12.9a. From the plot it is evident that the average value of the phase detector output diminishes with the increase of the open loop frequency error. This is similar in nature with that of a first order loop.

Fig. 12.9. The phase detector output vs frequency detuning characteristic of the delayless and delayed PLLs.
Let us now turn to the case when the loop incorporates a narrow band crystal filter in the IF stage. Here one can assume that the phase shift \( \phi \) is a linear function of the frequency error \( \delta \), and a relation [7],

\[
\phi = \frac{\delta}{\frac{3}{4} F K_F}
\]

(12.30)

may be taken with a fair degree of accuracy. Using this relation for \( \phi \) in conjunction with (12.76) the variation of the average value of the phase detector output with the open loop frequency error is shown in Fig. 12.9b. Looking at the graph one finds that there are values of \( \delta \) for which the phase detector output shows nulls. Obviously, if the loop jumps into one of these nulls, the pull-in voltage disappears, and there is a possibility that the loop remains at the frequency of the nulls. Out of these, nulls with positive slopes are stable whereas the nulls with negative slopes are unstable. Consider, for example, the point \( A \). If one moves away from \( A \), say, towards right, \( V_p \) becomes negative. This means that the control voltage \( V_p \) will be acting in such a way that it will push the frequency further, whereas if one considers the movement around \( B \) one finds that the control voltage acts in such a way as to oppose the cause to which it is due. Thus, once the loop gets into one of these nulls with positive slopes, it will stay there without showing any phase detector output. This gives an impression that the loop has achieved synchronization, which is really not true. Hence the name is false locking.

From (12.78) a interesting conclusion regarding the pull-in range of the PLL can be drawn. To begin with let us assume that the delay is absent and one may take \( \phi = 0 \). Note further that \( \delta / \frac{3}{4} F K_F \) gives the best angular frequency. The pull-in range occurs when \( \frac{3}{4} F K_F \) is infinite. Thus differentiating (12.78) with respect to \( \delta \) and setting

\[
\frac{d\phi}{d\delta} = \infty
\]

one finds that the pull-in range is approximately given by

\[
\delta_{\text{min}} = \sqrt{2 F K_F} \frac{\delta}{3}
\]

(12.81)

This value coincides with that derived by Richman (cf. 10.53).

Let us consider the case of the PLL incorporating a delay network having a delay time \( T \). Therefore, we assume
Phase Lock Theories and Applications

\[ \gamma = \Omega_1 T = \Omega_2 \]

We make this approximation since \( \Omega_1 \) is close to \( \Omega \). Therefore, using the same definition for calculating the pulling range, i.e., pulling occurs when \( \frac{\Omega_1}{\Omega} \to \infty \) one finds that the pull-in range with delay is given by the expression

\[ \Omega^2 = \frac{2 A K V_f}{\Omega_2} \cos \Omega T \]

(12.82)

A graphical method of solving the equation (12.82) for pull-in range is illustrated in Fig. 12.10. However, when \( T \) is not large, the locking range can be evaluated by expanding \( \cos \Omega T \), in which case it is given by

\[ \frac{\Omega}{\sqrt{2 A K V_f} F_0} = \frac{1}{\sqrt{1 + \frac{A K V_f}{F_0}}} \]

(12.83)

![Graphical method for solving the locking range equation of a PLL with delay.](image)

Fig. 12.10. Illustrating graphical method for solving the locking range equation of a PLL with delay.
However, the equation (12.82) further shows that the maximum value of $\Omega$ is related by the relation

$$\Omega_{\text{max}} T = \pi/2$$

i.e.,

$$4T_{\text{max}} = \pi \tag{12.84}$$

This result agrees with those of earlier workers [10, 11].

We have already seen that false locks appear because of the use of a narrowband IF filter with a number of poles. This causes a large phase shift over a small frequency difference. We have further seen that first stable false lock point (8) appears for $\omega' = 3\omega_0$. Thus a simple way to avoid false lock will be to use an IF filter that does not have more than two poles, which produces a minimum phase shift of 180° or $\pi$ radians. But this solution may not be acceptable for all practical situations. In cases where a narrowband IF filter with rectangular frequency response characteristic is required, it is advisable to use sweep techniques. It may be possible to sweep over the region of false lock by suitably adjusting the speed of sweeping. But this may pose a difficult problem in acquiring correct lock, so that even it is advisable to use a tunable IF phase locked loop [12]. Use of a phase modulator, the gain of which is controlled by the rectified output of the phase detector, helps eliminating of false locking in a heterodyned phase locked loop [13]. By properly choosing the filter functions [14] of a double phase locked loop [15], false locking can also be eliminated.

12.5 Multiplier PLL

For certain types of communication links, such as in the Apollo Unified S-Band mission, the spectral format of the modulating signal (Fig. 12.11) consists of a wide band video portion and a relatively high carrier subcarrier for telemetry, voice, etc. In the figure, for the sake of simplicity, we have depicted only one subcarrier along with the video portion. Thus the received signal for such types of modulating signal may be represented as

$$R(t) = \sqrt{2E} \sin (\omega_0 t + \theta(t)) + h(t) \tag{12.85}$$

where

$$\theta(t) = M(t) + \sum_{k=1}^{N} \phi_k (t) \sin (\omega_k t + \varphi_k) \tag{17.86}$$
where the first term on the right hand side of (12.86) represents the video signal and the second one denotes the subcarriers.

In order to demodulate a signal with the spectral format of Fig. 12.11, a conventional phase locked demodulator (PLD) with the frequency response characteristic, as shown by the dotted line of Fig. 12.11, can be used. Obviously, such a technique will allow the noise power in the unused band from $a_2$ to $a_3$ to creep into the loop and will contaminate the demodulated signal. An obvious way to improve the performance of a PLD for the type of modulating signal is to adjust the frequency response characteristic in such a way as to avoid the unused band in the way shown by the chained line of Fig. 12.11. This idea leads to the development of the so-called multiliter phase locked loop (MPLL) [40]. A typical realization of a MPLL for the spectral format of Fig. 12.11 is shown in Fig. 12.12. Representing the incoming signal for the Appello Unified S-band communication system as

$$v_t(t) = \sqrt{2} \sin [a_1 t + a_2 (t) + a_3 (t)] \quad (12.87)$$

where $a_1 (t)$ and $a_2 (t)$ are respectively due to the video signal and the subcarrier.

Now taking the output of the VCO as

$$v_s(t) = \sqrt{2} \cos [a_0 (t) + \beta v (t)] \quad (12.88)$$

the governing phase equation can be written as
Fig. 12.12. The multfilter PLL configuration.

\[ \frac{d\phi}{dt} = \Omega - AK [F_L(p) + F_H(p)] \sin \phi + \frac{d\phi_1}{dt} + \frac{d\phi_2}{dt} \]  
(12.89)

where
\[ \phi = (\omega_1 - \omega_0) t + \delta_s + \beta_1 + \alpha_2 - \beta_3 \]  
(12.90)

and
\[ \Omega = \omega_1 - \omega_0 \]

\( F_L(p) \) and \( F_H(p) \) are respectively the transfer functions of the low pass and band pass filters. Note that \( \beta_1 (t) \) and \( \beta_2 (t) \) respectively denote the phase modulation of the VCO due to the video signal and the subcarrier.

For the case of an in-tune carrier (\( \Omega = 0 \)), the closed loop transfer function of the linearised MFPLL can be written as (cf. 8.17)

\[ H(s) = \frac{\beta_1(s)}{\alpha_1(s)} = \frac{AK [F_L(s) + F_H(s)]}{s + AK [-F_L(s) + F_H(s)]} \]  
(12.91)

where
\[ \alpha_1(s) = s\beta_1(s) + \alpha_2(s) \]

and
\[ \beta_1(s) = \beta_1(s) + \beta_2(s) \]  
(12.92)

For an N-number of modulating signals, the above result for the transfer function may be generalised as
\[ H(\omega) = \frac{\sum_{i=1}^{n} F_i(\omega)}{s + AK \sum_{i=1}^{n} F_i(\omega)} \] (12.93)

Refer to the Fig. 12.11 and choose the frequency response characteristics of \(F_i(\omega)\) in such a way that the individual closed loop response \(AKF_i(\omega) + AKF_i(\omega)\) approaches zero at frequencies away from that of any other. This signifies the independence of the closed response of each filter. In such a case, one can approximate (12.91) as

\[ H(\omega) \approx \frac{AKF_i(\omega)}{s + AKF_i(\omega)} + \frac{AKF_i(\omega)}{s + AKF_i(\omega)} \] (12.94)

and hence (12.93) can also be approximated as

\[ H(\omega) \approx \sum_{i=1}^{n} \frac{AKF_i(\omega)}{s + AKF_i(\omega)} \] (12.95)

To study the stability of the MPFLL, we need to consider the location of the poles of the closed transfer function (cf. Chapter 8). Thus we need to find the roots of (cf. 12.91)

\[ s + AK[F_i(\omega) + F_d(\omega)] = 0 \] (12.96)

Taking the transfer functions of \(F_i(\omega)\) and \(F_d(\omega)\) as

\[ F_i(\omega) = 1 + ak \] (12.97)

and

\[ F_d(\omega) = \theta_1(\omega^2 + 2\omega_0\omega + \omega_0^2) \] (12.98)

Substitutions of these in (12.96), leads to the following biquadratic equation in \(\omega^2\)

\[ \omega^2 + 2\omega(aK + \omega_0) + \theta_1(\omega_0^2 + 2\omega_0\omega + \omega_0^2 + \omega_0^2) + \gamma_0 \omega_0 + 2\omega_0 ak_0 + aK\omega_0^2 = 0 \] (12.99)

For the system to be stable, the roots of (12.99) must have negative real part. The condition that guarantees location of the roots in the left half of the \(s\)-plane is obtained from Routh-Hurwitz theorem and is given by

\[ a_1 > 0, a_2 > 0, a_3 > 0 \text{ and } \] (12.100)

\[ a_i (a_i a_0 - a_0 a_i) - a_i^2 a_i > 0 \]

where

\[ a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4 = -0 \] (12.101)
Composing (12.100), (12.101) and (12.99) we find that the system will be stable provided

\[ a_2/d_2 - a_1/d_1 > 0 \]  
(12.102)

where

\[ a_1 = (w_0^2 + 2\omega_0\omega_1) \]
\[ a_2 = (w_0^2 + 2\omega_0\omega_1 + a_0\omega_1) \]
\[ a_0 = 2(K_0 + \omega_0^2) \]
and

\[ a = 2K_0w_1 \]

For the type of the low pass filter taken (12.99), the boosting range is infinite. But in practice, it is very difficult to realize a perfectly integrating filter, thus the transfer function of the low pass filter assumes the following form

\[ F_R(\rho) = \frac{h}{(1 + \epsilon_0 \rho)} \]  
(12.103)

in this case a knowledge of the boosting range of the MPPLL is essential and it can be estimated in the following way (cf. Chapter 10). Assume a solution of the equation

\[ \frac{d\rho}{dt} = \Omega - A_0 [F_S(\rho) + F_R(\rho)] \sin \varphi \]  
(12.104)

in the boosting condition, we assume a solution of (12.104) as

\[ \varphi = \omega t + m \sin \omega t \]  
(12.105)

and represent

\[ F(\rho) \exp (j\omega) = A_0 [F_S(\rho) + F_R(\rho)] \]  
(12.106)

Putting the solution of \( \varphi \) in (12.104) and using the method of harmonic balance as was done in Chapter 10, it is not difficult to show that

\[ \left[ \frac{H(\omega)}{m_0} \right] = \frac{2\omega_1 \sin \Omega}{(2\omega_0^2) - (\omega_0^2 + \omega_1^2)} \]  
(12.107)

and

\[ \Omega = \sqrt{1 + m_0^2 \sin^2 \varphi} \]  
(12.108)

Now in the locked state, since \( \varphi = \psi = 0 \), one finds that \( m_0 = \pm 1 \).

Thus (12.107) and (12.108) reduce to the following

\[ 0.9659F_S(\psi) = \psi \sqrt{2.202 - 603\psi^2} \]  
(12.109)
\[ L = \omega \left[ 1 + \frac{\cos \gamma}{2E(\phi)} \right] \quad (12.110) \]

Calculating the transitional value of the beat frequency from (12.109) and substituting it in (12.110) leads to the evaluation of the locking range.

12.5.1. Noise Bandwidth

If the signal is accompanied by narrowband Gaussian noise of one-sided spectral density \( N_p \), the net received signal can be written as (cf. Chapter 9)

\[ R(t) = \sqrt{2} \Delta \sin (\omega_0 t + n(t)) + n(t) \quad (12.111) \]

where

\[ n(t) = \sqrt{2} n(t) \cos \omega_0 t - \sqrt{2} \tilde{n}(t) \sin \omega_0 t \quad (12.112) \]

and

\[ \tilde{n}(t) = \sqrt{2} N_p(t) \cos [\omega_0 t + \tilde{n}(t)] - \sqrt{2} N_p(t) \sin (\omega_0 t + \tilde{n}(t)) \quad (12.113) \]

where

\[ N_p(t) = n(t) \cos \omega_0 t + \tilde{n}(t) \sin \omega_0 t \]

and

\[ N(t) = n(t) \cos \omega_0 t - n(t) \sin \omega_0 t \]

Thus taking the output of the VCO as

\[ v_c(t) = \sqrt{2} \cos (\omega_0 t + \phi_c(t)) \]

the loop equations can be written as

\[ \frac{dv_c}{dt} = \Omega - K \left[ \frac{\partial}{\partial \phi} + P_c(t) \right] (\Delta \sin \gamma + N(t)) \quad + \frac{\delta \phi_c}{dt} \quad (12.114) \]

where

\[ \phi = \Omega t + \phi_c(t) - \phi_0(t) \]

and

\[ N(t) = \sqrt{2} n(t) \cos \gamma - \sqrt{2} n(t) \sin \gamma \quad (12.115) \]

It is easily shown that

\[ v_c(t) = \frac{2n_0(t)}{\frac{K}{\sum_{i=1}^{N} F_i(t)} \sum_{i=1}^{N} F_i(t) - \sum_{i=1}^{N} P_i(t)} \quad (12.116) \]
Again
\[ h_d(t) = s_d(t) - y(t) \]
\[ = \frac{AK}{s + AK} f_d(t) - \frac{s_d(t) + N(t)}{A} \]  
(12.117)

Therefore, the variance of the phase error due to noise (assuming that the signal and noise are independent), is given by (cf. 9.37)
\[ \sigma_{\phi_{\text{noise}}}^2 = \frac{N_0}{4\pi A^2} \int \frac{AK}{s + AK} f_d(t) f_d(t) \, ds \] 
Hence
\[ B_n = \frac{1}{4\pi} \int \frac{AK}{s + AK} f_d(t) \, ds + \frac{1}{4\pi} \int \frac{AK F_0(t)}{s + AK} F_0(t) \, ds \] 
(12.119)

since the closed loop responses of the lowpass and the bandpass filters are non over-lapping (12.119) can be written as
\[ B_n = \frac{1}{4\pi} \int \frac{AK f_d(t)}{s + AK} \, ds + \frac{1}{4\pi} \int \frac{AK F_0(t)}{s + AK} F_0(t) \, ds \] 
(12.120)

Taking \( F_0(t) = \frac{1}{s + s_0} \) and putting \( \sigma_{\phi_{\text{noise}}}^2 = AK/T \rho \) and \( 2\sigma^2_{\text{noise}} = AKT \rho \), it is easy to show that
\[ \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{AK f_d(t)}{s + AK} \, ds = \frac{2\pi \rho}{\pi} (\xi_0 + \xi_0) \] 
(12.121)

Now taking
\[ F_d(t) = \frac{AK f_d(t)}{s + 2\rho \xi_0 s + \rho^2} \]  
(12.121)

It can be shown that
\[ \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{AK f_d(t)}{s + 2\rho \xi_0 s + \rho^2} \, ds = \frac{4\pi K \rho^2}{\rho^2 \xi_0^2 + AK} \]
Therefore,
\[ b_n = \frac{\theta_n}{2} \left( \frac{1}{K} + \frac{1}{K} \right) + \frac{(4\Delta f)^2}{2K} \]  
\[ \text{(12.122)} \]

12.6 Multiloop Multifilter PLL

In order to improve the tracking and locking characteristics of the multifilter PLL, a multiloop MFPLL has been proposed. This is shown in Fig. 12.13. It consists of two phase-locked loops. The inner loop incorporates the bandpass filters, whereas the outer one incorporates lowpass filters. For the sake of simplicity, only one bandpass filter and one lowpass filter have been depicted in the figure. The inner loop does not have dc gain as such injection synchronization is used to serve the purpose of acquisition. To derive the system equation, we assume that the outputs of the VCO-1 and VCO-2 are respectively given by

![Diagram of Multiloop Multifilter PLL](image)

\[ r_1(t) = \sqrt{2} \cos (\omega_c t + \theta(t)) \] \[ \text{(12.123)} \]
\[ r_2(t) = \sqrt{2} \cos (\omega_a t + \theta(t)) \] \[ \text{(12.124)} \]

Putting
\[ \varphi = (\omega_a - \omega_c) t + \phi(t) - \psi(t) \] \[ \text{(12.125)} \]
and assuming the incoming signal as in (12.87), i.e.,
\[ v(t) = 24 \sin (\omega_0 t + \phi(t)) \]  

(12.126)

It is not difficult to show that

\[ \frac{db}{dt} = A K F_j (p) \sin \varrho \]  

(12.127)

\[ \frac{dc}{dt} = A[K F_j(p) + K_d] \sin \varrho \]  

(12.128)

where

\[ K_a = \frac{x}{\omega_0} \frac{1}{Q} \]  

(12.129)

B denotes the amplitude of the oscillator of VCO-1, and \( x \) is the fraction of the input used for injection synchronizing the VCO-1 and \( \omega_0 \) is the free-running frequency of the VCO-1. Thus from (12.123) to (12.129), it is easily shown that

\[ \frac{d\varrho}{dt} = \Omega - A[K_a + K F_j(p) + K F_j(p)] \sin \varrho + \frac{db}{dt} + \frac{dc}{dt} \]  

(12.130)

Note that for the sake of simplicity, sensitivities of the VCO-1 and VCO-2 have been taken to be identical.

12.6.1. Frequency Response Characteristics

For the purpose of finding the frequency response characteristics, we assume that

\[ \Omega = 0 \] (in-tune carrier)

\[ \sin \varrho \approx \varrho \] (small phase error)

Thus from (12.125) to (12.129), we find

\[ G_0(\varrho) = \frac{AF_j(\varrho)}{\omega_0(\varrho)} = \frac{AF_j(p)}{\omega_0} \]  

(12.131)

and

\[ G_0(\varrho) = \frac{AF_j(\varrho) s(\varrho)}{\omega_0(\varrho)} = \frac{AF_j(p) s(p)}{\omega_0} \]  

(12.132)

Taking

\[ F_d(\varrho) = 1 + a \]
and

\[ F(\theta) = \frac{\theta}{\theta^2 + 2\omega_n\omega \theta + \omega_n^2} \]

It is easily shown that

\[ G_2(\theta) = \frac{(s + a) (\theta^2 + 2\omega_n\omega \theta + \omega_n^2)}{AK^2 + (\theta^2 + 2\omega_n\omega \theta + \omega_n^2)[(s + \omega_0)(\theta^2 + 2\omega_n\omega + AK) + \omega_n^2]} \]  
(12.133)

and

\[ G_3(\theta) = \frac{s^2}{AK^2 + (\theta^2 + 2\omega_n\omega \theta + \omega_n^2)[(s + \omega_0)(\theta^2 + 2\omega_n\omega + AK) + \omega_n^2]} \]  
(12.134)

where,

\[ 2\omega_n\omega = 2AK + AK_0 \]
(12.135)

\[ \omega_0 = a AK \]

Remembering that the centre frequency of the bandpass signal is large compared to the bandwidth of the video signal, it is not difficult to show that the transfer function \( G_2(\theta) \) in the video band reduces to

\[ \frac{s + a}{s + AK + AK_0} \]

If

\( AK_0 \ll AK \) one finds that

\[ G_2(\theta) \approx \frac{F(\theta)}{s + AK + AK_0} \]  
(12.136)

Similarly, around the centre frequency of the bandpass filter,

\[ G_3(\theta) \approx \frac{s}{s + (AK + AK_0) s + \omega_n^2} \]  
(12.137)

Thus it is seen that \( G(\theta) \) and \( G_3(\theta) \) has the lowpass and bandpass characters. It is interesting to note that since the lowpass and bandpass signals are fed to two different VCO's, the nonlinear distortion arising out of VCO nonlinearity will be less. The locking range characteristics of the multistage PLL and multiloop NEPLL are shown in Table 12.1.
TABLE 12.1
Locking Range Characteristics

<table>
<thead>
<tr>
<th>System</th>
<th>Type of filter</th>
<th>Low pass</th>
<th>Locking ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mulloop</td>
<td></td>
<td>1</td>
<td>2.25</td>
</tr>
<tr>
<td>MFPLL</td>
<td></td>
<td>1</td>
<td>1.65</td>
</tr>
</tbody>
</table>

REFERENCES

Hase Lock Theories and Applications


We have seen in Chapter 7 that the noise-free performance of a PLL depends on the loop natural frequency \( \omega_n \) and the damping factor \( \xi \), which in turn depend on the open loop gain \( A_K \) and the time constant of the loop filter. Thus any change in any of these parameters will affect the performance of the loop. This is particularly true when the incoming signal is accompanied by additive random disturbances.

In this case also we have noted in Chapter 7 that the value of the loop natural frequency for optimum operation of a PLL depends on the input carrier-to-noise power ratio. Thus it is apparent that a loop, which gives optimum performance for a particular value of the input CNR, is not definitely optimum for other values of the input CNR. Thus in order to cause the PLL to adapt itself to the varying conditions of the input CNR, it is often preceded by a bandwidth limiter (BPL). It is thus interesting first to learn the properties of a BPL and then to see how incorporation of a BPL helps in achieving the said purpose.

### 13.1 Transform Method of Analysis for a BPL

In this section, we will follow Davenport's classic analysis [1] on a bandwidth limiter. A typical BPL configuration is shown in Fig. 13.1. Here we will only briefly review the method. Let us suppose that the nonlinearity of the limiter characteristic is represented by \( f(e) \). Then the bilateral Laplace transform of the nonlinearity can be written as...
\[ F(z) = \int_{-\infty}^{\infty} f(x) e^{i\omega t} \, dx \]

Thus we have taken

\[ f_+ (x) = f(x) \quad x > 0 \]

\[ f_-(x) = 0 \quad x < 0 \]

and

\[ f_+(x) = f_-(x) \quad x \leq 0 \]

\[ f_+(x) = 0 \quad x > 0 \]

If the transfer characteristic of the non-linear element is an odd function of its arguments, then \[ F_+ (\omega) = -F_- (\omega) \]

\[ F_+ (\omega) = -F_- (\omega) \quad F_+ (\omega) \text{ and } F_- (\omega) \text{ are analytic on the right half and left half planes respectively.} \]

If the transfer characteristic of the non-linear element is an odd function of its arguments, then \[ F_+ (\omega) = -F_- (\omega) \]

Thus transform of \( f_+ (x) \) \( F_+ (\omega) \) can be found by taking the usual one-sided Laplace transform of \( f_+ (x) \) and then replacing \( x \) by \(-x\). Thus

\[ f(x) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} F(\omega) e^{\omega t} \, d\omega \]
\[
\delta(t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} F_\delta(s) e^{st} ds + \frac{1}{2\pi j} \int_{-\infty}^{\infty} F_- (s) e^{st} ds
\]

(3 \geq 0)
\[\gamma > 0\]

(13.4)

For a hard limiter, represented by the characteristic
\[y = \text{Lsgn}(x)\]

one finds that
\[f_\delta(t) = -f_- (t)\]

i.e.,
\[F_\delta(s) = \frac{L}{s}\]

But have
\[F_\delta(s) = \frac{L}{\gamma} \]

i.e.,
\[F_\delta(s) = \frac{L}{\gamma}\]

Therefore,
\[F_\delta(s) = \frac{2L}{\gamma}\]

(13.5)

Now if the transfer characteristic is linear, i.e.,
\[f_\delta(t) = \gamma t\]

then
\[F_\delta(s) = -f_- (t)\]

and
\[F_\delta(s) = -f_- (s)\]

Using (13.2) one finds
\[F_\delta(s) = \left[\frac{L}{s}\right]^2\]

Thus
\[\gamma = \lim_{\gamma \to \infty} \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{1}{s} e^{st} ds + \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{1}{s} e^{st} ds\]

\[= \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{1}{s} e^{st} ds\]

(13.6)
where is the integral along a closed circle with the origin as the centre. Putting \( z = j \omega \) in (13.8) becomes

\[
x = \frac{1}{\omega} \mathcal{F}^{-1} \left[ \frac{1}{(1+j\omega)^2} \right] d\omega
\]

(13.7)

Let us now suppose that the input \( q(t) \) is a sum of two stationary random variables \( x_1(t) \) and \( x_2(t) \) which are statistically independent of each other. Now if we assume that the true output of the nonlinear element is represented by the approximate output \( g_0 x_1(t) \) \( + g_2 x_2(t) \), it is not difficult to show that the mean square difference between the actual output and the approximate output will be minimum, provided

\[
R_{xx}(\tau) = g_0 R_{xx} x_1(\tau)
\]

(13.8)

and

\[
R_{xx}(\tau) = g_2 R_{xx} x_2(\tau)
\]

(13.9)

Here \( g_0 \) and \( g_2 \) represent respectively the noise and signal gains of the nonlinear element. The above results are obtained by differentiating the mean square error with respect to \( g_0 \) and \( g_2 \) and putting them equal to zero. It is to be noted that \( R_{xx}(\tau) \) represents the cross-correlation function of \( x_1(t) \) and \( x_2(t) \) and \( R_{xx}(\tau) \) represents the auto-correlation function of \( x(t) \). The above cross-correlation function of (13.8) and (13.9) are defined as

\[
R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) x(t+\tau) dt
\]

(13.10)

\[
R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_2(t) y(t+\tau) dt
\]

(13.11)

\[
R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} y(t) y(t+\tau) dt
\]

(13.12)

and

\[
R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_1(t) x_2(t+\tau) dt
\]

(13.13)

Therefore using (13.7) in conjunction with (13.2), (13.10) and (13.11) one finds that
\[ R_{\text{anr}} (\tau) = \frac{1}{4\pi} \int_0^\pi (\beta \omega)^2 \, d\omega \int f(\tau) \, \mathcal{Y}_2 (u, \tau; \gamma) \, \mathcal{V}_2 (v, \tau) \, d\theta \]  
(13.14)

and

\[ R_{\text{anr}} (\tau) = \frac{1}{4\pi} \int_0^\pi (\beta \omega)^2 \, d\omega \int f(\tau) \, \mathcal{Y}_3 (u, \tau; \gamma) \, \mathcal{V}_4 (v, \tau) \, d\theta \]  
(13.15)

where

\[ \mathcal{Y}_2 (u; \tau; \gamma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T \exp \left\{ j\omega \tau + j\omega \gamma (t + \tau) \right\} \, dt \]  
(13.16)

\[ \mathcal{Y}_4 (u; \tau; \gamma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T \exp \left\{ j\omega \tau + j\omega \gamma (t + \tau) \right\} \, dt \]  
(13.17)

Now since \( x(t) \) and \( x(t) \) are independent stationary processes, it is easily shown that

\[ \mathcal{W}_f (u; \tau; \gamma) = M_f (u, \tau; \gamma) \]  
(13.18)

and

\[ \mathcal{W}_f (u; \tau; \gamma) = M_f (u, \tau; \gamma) \]  
(13.19)

Here \( M_f (u, v; \gamma) \) is the joint characteristic function of the two random variables \( x(t) \) and \( x(t + \tau) \); \( M_f (u, v; \gamma) \) is the joint characteristic function of two random variables \( \alpha(t) \) and \( \alpha(t + \tau) \). \( M_f (u; \gamma) \) and \( M_f (v; \gamma) \) are respectively the characteristic functions of the random variables \( x(t) \) and \( x(t) \). These functions are defined as:

\[ M_f (u; \gamma) = \exp \left\{ j\omega u + j\omega \gamma (t + \tau) \right\} \]  
(13.20)

\[ M_f (v; \gamma) = \exp \left\{ j\omega v + j\omega \gamma (t + \tau) \right\} \]  
(13.21)

\[ M_f (u; \gamma) = \exp \{ j\omega u \} \]  
(13.22)

and

\[ M_f (v; \gamma) = \exp \{ j\omega v \} \]  
(13.23)

where the symbol \( \gamma \) indicates the average value; with these general remarks let us now consider the following cases.

Case I: Input is a sum of sinusoidal wave and a Gaussian random noise
This can be written as

\[ Z(t) = E(t) \cos (bt - \gamma(t)) \cos (bt - \gamma(t)) \cos (bt - \gamma(t)) \]

\[ = V \cos \psi \]

(13.25a)

and

\[ \gamma(t) = \theta + \tan^{-1} \left( \frac{m_n}{I + n} \right) \]

(13.27)

Let us now calculate the output of the limiter (hard). Using (13.4) and (13.25a) and remembering that the hard limiter characteristic is symmetrical, we write

\[ y = f(x) = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} F_\omega(t) \exp (i \psi \cos \psi) \psi \cos \psi \]

(13.28)

Again using the Jacobi-Thuer formula [4],

\[ \exp (i \psi \cos \psi) = \sum_{m=-\infty}^{\infty} \cos m \psi \cos n \psi \]

(13.29)

where \( F_\omega(t) \) is the modified Bessel's function of order \( m \) and argument \( \omega \). Hence we find from (13.28) and (13.29)

\[ y = \int_{-\pi/2}^{\pi/2} \sum_{m=-\infty}^{\infty} \cos m \psi \cos n \psi \int_{-\pi/2}^{\pi/2} \frac{F_\omega(t) \psi \cos \psi}{I \cos \psi} \psi \cos \psi \]

(13.30)

Putting \( \psi = u + \epsilon = \delta \psi \) one gets

\[ y' = \int_{-\pi/2}^{\pi/2} \sum_{m=-\infty}^{\infty} \cos m \psi \cos n \psi \int_{-\pi/2}^{\pi/2} \frac{F_\omega(t) \psi \cos \psi}{I \cos \psi} \psi \cos \psi \]

(13.31)
\[
\text{use } \int f(x) \, dx = \frac{b-a}{x} f(x) - J_{a+1}(x) + J_{a+1}(x) \\
\text{i.e.,}
\]

\[
y = \frac{4L}{\pi} \sum_{n=0}^{\infty} \left( -1 \right)^n \cos \left( m + 1 \right) (\beta t + \gamma) (13.32)
\]

Therefore, the nth zonal power output is \( P_n = (\delta / \omega n) L^2 \) (13.33)

Referring to (13.24) one finds that

\[
x_2(t) = 2A \cos (\beta t + \theta) \\
\text{Therefore,}
\]

\[
\sin x(t) = x_2(t) = 2A \sqrt{2 \sqrt{a^2 + v^2 + \beta^2} \cos \beta t + \sin \beta} \sin \beta
\]

\[
\psi_i = \beta t + \theta t + \arctan \frac{\sqrt{2} \sin \beta}{u + v \cos \beta}
\]

Therefore, using the method of Rice

\[
M_{11}(u, v; r) = \langle \exp (i u \omega_0 t) + x r(t + \gamma) \rangle = J_0(\sqrt{2} A \cos \beta t + \sin \beta) r h(t)
\]

which may be written as (4)

\[
M_{11}(u, v; r) = \sum_{n=0}^{\infty} \alpha_n (J_{2n+1} \psi(t) + J_{2n+1} \psi(t))
\]

Similarly

\[
M_r(u, v; r) = \exp \left( -\frac{\beta^2 + v^2}{2} \right) \frac{\pi}{\pi} \frac{\gamma}{\gamma}
\]

and

\[
M_0(u, v; r) = \exp \left( -\frac{\beta^2 + v^2}{2} \right)
\]

Thus

\[
\psi(t, u, v; r) = \sum_{n=0}^{\infty} \alpha_n (J_{2n+1} \psi(t) + J_{2n+1} \psi(t)) \exp \left( \frac{-\beta \sqrt{2} A \cos \beta t}{2} \right)
\]

(13.34)

(13.35)

(13.36)

(13.37)

(13.38)

(13.39)
\[ R_{mn}(\alpha) = i \frac{1}{2\pi} \int \overline{\chi(\rho)} R_{\rho \alpha} \exp \left( -\frac{i\alpha}{2} \rho^2 \right) d\rho \cdot \cos \beta \]  
\[ (13.42) \]

where
\[ \alpha_1 = \frac{1}{\sqrt{2}} \int \check{F}(j\rho) \overline{\check{I}_2(j\sqrt{2}\rho)} \exp \left( -\frac{i\rho^2}{2} \right) d\rho \]  
\[ (13.43) \]

Using the result of Davenport and Root [5] one finds that
\[ R_{mn}(\alpha) = \sqrt{2} \alpha n_{a} \cos \beta \]  
\[ (13.44) \]

Further referring (13.8) one finds
\[ R_{mn}(\alpha) = \sqrt{2} \alpha n_{a} \cos \beta \]  
\[ (13.45) \]

Hence from (13.44), (13.45) and (13.48) one finds
\[ g_2 = \frac{1}{4\pi} \int \overline{F}(j\rho) \rho \overline{\check{I}_2(j\sqrt{2}\rho)} \exp \left( -\frac{i\rho^2}{2} \right) d\rho \]  
\[ (13.46) \]

Similarly using (13.41) and (13.45) one finds
\[ R_{mn} = \alpha_1 R_{mn}(\alpha) \]  
\[ (13.47) \]

where
\[ \alpha_1 = \frac{1}{\sqrt{2}} \int \overline{\chi(\rho)} \overline{\check{I}_2(j\sqrt{2}\rho)} \exp \left( -\frac{i\rho^2}{2} \right) d\rho \]  
\[ (13.48) \]

Comparing this with (13.9) one gets
\[ g_2 = \frac{1}{4\pi} \int \overline{F}(j\rho) \rho \overline{\check{I}_2(j\sqrt{2}\rho)} \exp \left( -\frac{i\rho^2}{2} \right) d\rho \cdot \alpha_1 \]  
\[ (13.49) \]

For a hard limiter
\[ F(j\rho) = 2\rho \]  
\[ \check{F}(j\rho) = \sqrt{2\pi} \sum \delta \]  
\[ \check{I}_2(j\sqrt{2}\rho) = \sqrt{2\pi} \sum \delta \]
Therefore, 

\[ g_s = \frac{\sqrt{2}}{A} \frac{L_A}{2} \int \frac{b}{r} J_b (r) \exp \left( -\frac{z^2}{2} \right) dv \]

Noting that 

\[ I_b (r) = J_b (r) \]

Thus 

\[ g_s = \frac{4L_B}{2\pi} \frac{L_A}{2\pi} \int \frac{b}{r} J_b (r) \exp \left( -\frac{z^2}{2} \right) dv \]

\[ \text{Use} \quad J_b (r_0) \cdot r^m \cdot e^{-\gamma r^2} \cdot dz = \frac{\Gamma(\frac{m+n}{2})}{2\pi^{\frac{n}{2}} \Gamma(\frac{n}{2})} \left( \frac{\gamma}{\alpha} \right)^\frac{n}{2} \]

\[ J_b (r_0) \left( \frac{r_0 \gamma}{\alpha} \right)^\frac{n}{2} \int \frac{b}{r} J_b (r) \exp \left( -\frac{z^2}{2} \right) dv = \frac{\Gamma(\frac{m+n}{2})}{2\pi^{\frac{n}{2}} \Gamma(\frac{n}{2})} \left( \frac{\gamma}{\alpha} \right)^\frac{n}{2} \int J_b (r) \exp \left( -\frac{z^2}{2} \right) dv \]

\[ \text{Real m + n > 0 arg b > n/2} \]

\[ g_s = \frac{\sqrt{2}}{A} \frac{L_A}{2} \int J_b (r) \exp \left( -\frac{z^2}{2} \right) dv = \frac{\sqrt{2}}{\alpha} \int J_b (r) \exp \left( -\frac{z^2}{2} \right) dv \]

\[ \int J_b (r) \exp \left( -\frac{z^2}{2} \right) dv = \frac{\sqrt{\pi}}{\alpha} \int J_b (r) \exp \left( -\frac{z^2}{2} \right) dv \]

\[ g_s = \frac{\sqrt{2}}{A} \frac{L_A}{2} \int J_b (r) \exp \left( -\frac{z^2}{2} \right) dv \]

where \[ \gamma = \frac{\sqrt{\alpha}}{A} = (CNR)_b \]

Therefore, 

\[ g_s = \frac{\sqrt{2} \pi L_A}{4A} \int J_b (\sqrt{2} \alpha) \exp \left( -\frac{z^2}{2} \right) dv \]

\[ \text{Similarly the noise gain is given by} \]

\[ g_s = \frac{4L_B}{2\pi} \frac{L_A}{2\pi} \int J_b (\sqrt{2} \alpha) \exp \left( -\frac{z^2}{2} \right) dv \]

\[ = \frac{2\pi}{\sqrt{2}\alpha} \int J_b (\sqrt{1} \gamma) \exp \left( -\frac{z^2}{2} \right) dv \]

\[ \text{Note that this expression is valid for high value of the carrier to noise ratio.} \]

\[ (13, 59) \]

\[ (12, 51) \]
\[ a = (\gamma, A)^{1/2} P_x \]
\[ = \sqrt{\frac{2\pi}{2}} \exp(-\phi/2) [I_0(\phi/2) + I_1(\phi/2)] \quad (13.52) \]

Obviously, the total noise power output in the first zone (cf. 3.33)
\[ N_0 = \frac{8\pi^2}{n^2} - S_0 \]
where
\[ S_0 = \frac{4\pi^2}{n^2} \exp(-\phi) [I_0(\phi/2) + I_1(\phi/2)] \quad (13.33) \]

Therefore, when \( \phi \to \infty \), one gets by putting
\[ I_0(\phi/2) = \frac{\exp(\phi/2)}{\pi} \left( 1 + \frac{1}{4\phi} \right) \]
\[ I_1(\phi/2) = \frac{\exp(\phi/2)}{\pi} \left( 1 - \frac{3}{4\phi} \right) \]
\[ [S_0]_{\phi \to \infty} = \frac{2\pi^2}{n^2} \left( 2 - \frac{3}{4\phi} \right) \frac{1}{4\phi} \left( 1 - \frac{1}{2\phi} \right) \]
and
\[ [N_0]_{\phi \to \infty} = \frac{2\pi^2}{n^2} \frac{4\pi^2}{n^2} \]

That is,
\[ \left( \frac{S_0}{N_0} \right)_{\phi \to \infty} = 2 \quad (13.54) \]

when \( \phi \to 0 \) one put \( I_0(\phi/2) = 1 \) and \( I_1(\phi/2) = 0 \).
That
\[ [S_0]_{\phi \to 0} = \frac{2\pi^2}{n^2} e^{\phi} \]
\[ [N_0]_{\phi \to 0} = \frac{8\pi^2}{n^2} \]
Thus
\[ \left( \frac{S_0}{N_0} \right)_{\phi \to 0} = \frac{\pi}{4^\phi} \]
For high values of the input CNR’s the output noise power can be calculated by using the expression for $g_s$ and (13.24). Thus the noise power output is given by

$$N_o = \frac{4L^2}{\pi} \rho_o^2 (\frac{1}{1 - \rho})$$

For large values of $\rho$

$$sF_\text{in} (\theta; 1; -\rho) = e^{\theta} F_\text{in} (\theta/2) \approx \frac{1}{2\rho}$$

Thus

$$N_o = \frac{4L^2}{\pi} \cdot \frac{1}{2\rho}$$

The variation of the output signal-to-noise ratio with the input carrier-to-noise power ratio is shown in Fig. 13.2.

![Graph showing the output signal-to-noise ratio with input carrier-to-noise power ratio.](image)

**Fig. 13.2** The output SNR/CNR versus input SNR characteristic of a binary phase detector.

**Case II: Input is simply a sinusoidal wave**

Let $Z = \sqrt{\frac{A}{2\pi}} \cos (\theta - \phi)$

Therefore, the signal gain $g_s$ is given by (cf. 13.30) ($\rho^2 \approx 0$)

$$g_s \approx 2 \cdot \frac{1}{2\pi} \int \rho(\theta) I_1 (\sqrt{2\pi} A) d\theta$$

(13.36)
Now
\[ I_l \left( j\sqrt{2}A_t \right) = jA_t \left( j\sqrt{2}A_t \right) \]
\[ = \frac{j}{2\pi} \int_0^{2\pi} \exp \left( j\sqrt{2}A_t \sin \varphi - \varphi \right) d\varphi \]  
(13.57)

Thus
\[ g^* = \frac{2}{A} \int_0^{\pi} \exp \left( -j\varphi \right) \left( \frac{1}{2\pi} \int f(v) \exp \left( j\sqrt{2}A_t \sin \varphi \right) dv \right) d\varphi \]

Comparing with (13.4) one finds that:
\[ g^* = \frac{\sqrt{2}}{A} \int_0^{\pi} \exp \left( -j\varphi \right) f(\sqrt{2}A_t \sin \varphi) d\varphi \]

That is
\[ g^* = \frac{\sqrt{2}}{A} \int_0^{\pi} f(\sqrt{2}A_t \sin \varphi) \sin \varphi d\varphi \]
(13.58)

\[ = \frac{\sqrt{2}}{A} \int_0^{\pi} (\sqrt{2}A_t \sin \varphi) \cos \varphi d\varphi \]

This is the well known complex form of the describing function (6).

Case III: Input is a stationary random noise
In this case signal output will be zero and the noise gain is obtained from (13.49) as
\[ g_* = \frac{1}{2\pi} \int f(v) h(v) \exp \left( -\frac{v^2}{2} \right) dv \]  
(13.59)

Noting that
\[ \exp \left( -\frac{v^2}{2} \right) = \int \exp \left( j\nu_0 \right) p(\nu) d\nu \]  
(13.60)

where \( p(\nu) \) is the probability density distribution function of a gaussian random process and is given by
\[
p(\omega) = \frac{1}{\sqrt{2\pi\Delta}} \exp \left( -\frac{\omega^2}{2\Delta^2} \right)
\]

Therefore, from (17.59) and (13.60) one writes

\[
\begin{align*}
\xi_n &= \int_{-\infty}^{\infty} P(\omega) \left( \frac{1}{2\pi} \int F(j\omega) \left( j\omega \right) \exp \left( j\omega \right) d\omega \right) d\omega \\
\therefore \xi_n &= \int_{-\infty}^{\infty} P(\omega) \left( \frac{1}{2\pi} \int F(j\omega) \exp \left( j\omega \right) d\omega \right) d\omega
\end{align*}
\]

i.e.,

\[
\xi = \int_{-\infty}^{\infty} F(\omega) p(\omega) d\omega
\]

(13.61)

Thus is the standard form of the noise gain of a nonlinearity.

13.3 PLL Proceed by a BPL

Let us consider the PLL proceed by a BPL as shown in Fig. 13.3. Further assume that the incoming signal is accompanied by a white Gaussian random noise. The signal plus noise is first passed through the IF filter of bandwidth B and then it is passed through a BPL, which limits the net signal at a height of \( L = b \). Therefore, the components of the signal voltage fed at the input to the phase detector is obtained from (13.50) and (13.24).

![Diagram](image.png)

Fig. 13.3. The PLL proceed by a bandpass limiter.
\[ S(t) = \sqrt{2} \cdot \frac{\sqrt{2\pi}}{\pi} \exp \left( - \frac{\pi}{2} f_0 t_0 / (2) \right) \cos \left( \omega_0 t + \theta \right) \] (13.52)

where \( \sqrt{2} \cdot \sqrt{4\pi} \cos (\omega_0 t + \theta) \) is the signal component at the input to the loop, which can be written as

\[ S(t) = \sqrt{2} \cdot \sqrt{4\pi} \cos \left( \omega_0 t + \theta \right) \] (13.62a)

Putting

\[ \eta = \frac{a \sqrt{4\pi}}{\pi} \] (13.63)

one finds that the transfer function of the PLL is given by

\[ H_f(t) = \frac{-\sqrt{2} \cdot \sqrt{4\pi} \eta}{x + \sqrt{2} \cdot \sqrt{4\pi} \eta} \] (13.64)

Therefore, the noise bandwidth of the PLL, preceded by a BPL and incorporating an integrating filter, is given by (cf. 9.76)

\[ B_n = \frac{\eta \cdot \left( t_{41} + \frac{1}{4t_{41}} \right)}{2} \] (13.65)

where,

\[ \frac{\omega_{26}^2}{\pi} = \frac{2}{\sqrt{4\pi}} \cdot \frac{a \sqrt{\pi}}{T_4} \] (13.66)

\[ \frac{\omega_{41}^2}{\pi} = \frac{2}{\sqrt{4\pi}} \cdot \frac{a \sqrt{\pi} T_4}{T_6} \]

\[ \frac{\xi_{41}^2}{T_6} = 0.225 \cdot \frac{a \sqrt{\pi} T_4}{T_6} \]

The minimum value of \( B_n \) now becomes

\[ (B_n)_{\text{min}} = 0.47 \sqrt{\pi} \frac{a \sqrt{\pi}}{T_1} \] (13.67)

which is a function of the input carrier-to-noise ratio \( \rho \), because of the presence of the signal amplitude suppression factor \( a \). It is known that the value of \( \omega_{41}^2 \), for which the minimum value of \( B_n \) appears is 0.5. That is,

\[ 0.25 - 0.225 \cdot \frac{a \sqrt{\pi} T_4}{T_6} \] (13.68)

which clearly shows that the minimum value of \( B_n \) will occur only at a particular value of \( a \), i.e., for a particular value of \( \rho \).
Let us now look back into the case when the PLL is not preceded by a BPL. In this case, the input CINR is given by

$$\varphi = \frac{A^2}{N_0B}$$

(13.69)

and the corresponding phase error variance is given by (cf. 9.67)

$$\frac{\sigma^2}{\varphi} = \frac{A^2}{B} \frac{1}{\sigma_o^2} = \frac{a}{B}$$

(13.70)

This indicates that the loop phase error variance is inversely proportional to the input carrier-to-noise ratio. The stability of a bandpass limiter is possibly best explained by considering the following design problem.

It is required to design a phase locked loop that will faithfully track a carrier under the following situations:

(i) when the input carrier to noise ratio is measured over a band of 100 KHz is 20 dB, and

(ii) when the input carrier to noise ratio is 10 dB but the carrier is subjected to a linear frequency drift of 50 Hz/sec.

Now by a linear analysis we mean that the standard deviation of the noise phase error remains within about 0.1 rad. Let us now suppose that we are considering a second order PLL with an integrating filter. Using (13.70) one finds that

$$\frac{(0.1)^2}{\varphi} = \frac{1}{\sigma_o^2} \frac{A^2}{B}$$

(13.71)

where

$$\sigma = \frac{a}{\sqrt{B}} = 10^{-4}$$

$$B = 100 \text{ KHz}$$

Thus

$$\frac{A}{B} = 10^{-4}$$

(13.72)

In,

$$\sigma_o = 10 \text{ Hz}$$

Since the loop is used to track a noisy signal it is expected that the loop parameters are adjusted for the minimum noise band with condition. Therefore the required loop natural frequency under this condition is
\( \omega_{\text{NL}} = 20 \text{ rad/sec.} \) (13.73)

In the second case, the frequency error produced is \( \frac{2\pi R}{\tau} \), i.e.,
\[
2\pi \times 50 \text{ rad/sec} = 0.1
\]

or
\( \omega_{\text{NL}} = 56.05 \text{ rad/sec} \) (13.74)

Thus it is apparent that the same PLL will not be able to track the signals under optimum condition. Now suppose that the loop is preceded by a BPF with a bandwidth of \( B_p \). In this case, if \( \omega \) is the CNR at the input to the limiter, i.e., after the IF filter of bandwidth \( B_p \), the ratio of \( \omega_{\text{NL}} \) and \( \omega \) will be given by
\[
\frac{\omega_{\text{NL}}}{\omega} = \frac{\omega_{\text{NL}}(\omega_p)}{\omega(\omega_p)} = \frac{56.05}{20}
\]

i.e.,
\[
\frac{\omega_{\text{NL}}}{\omega} = 2.8\text{ at}
\]

where
\[
b = \frac{\omega^2}{N_v B_p}, \quad A^2 \frac{B}{B_p} = \frac{\omega}{\omega_p}
\]

(13.75)

(13.76)

(13.77)

Now let us refer to the plot of \( v_\phi \) as shown in Fig. 13.4. From

Fig.13.4. Variation of the limiter suppression (\( v_\phi \)) with the input CNR.
Using of a Bandpass Limiter

In the above relation we construct the following Table.

<table>
<thead>
<tr>
<th>$H_0/H_p$</th>
<th>$P_1$</th>
<th>$a_1$</th>
<th>$P_2$</th>
<th>$a_2$</th>
<th>$a_3/H_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01</td>
<td>0.08</td>
<td>1</td>
<td>0.97</td>
<td>11.02</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.273</td>
<td>10</td>
<td>0.91</td>
<td>5.58</td>
</tr>
<tr>
<td>5</td>
<td>0.05</td>
<td>0.19</td>
<td>5</td>
<td>0.98</td>
<td>5.15</td>
</tr>
<tr>
<td>4</td>
<td>0.04</td>
<td>0.17</td>
<td>4</td>
<td>0.94</td>
<td>5.76</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>0.15</td>
<td>3</td>
<td>0.98</td>
<td>6.23</td>
</tr>
</tbody>
</table>

Thus it is seen that if $1$ BPL of $H_0/H_p = 3$ i.e., a bandwidth of 33.3 KHz issued before the PLL, then the PLL will track both the signals without crossing the limit of standard noise phase error of 0.1 radian.

**REFERENCES**

PHASE DETECTOR RESPONSE TO NOISY SIGNALS

We have seen that the response characteristic of a phase detector completely defines the noise-free performance of a phase locked loop. Obviously, in a noisy environment the response characteristic of a phase detector to noisy signals will govern the behaviour of the loop. Hence the noise handling capacity of a phase detector is a very important information required for the design of a phase locked receiver system. In this chapter, we will discuss the behaviour of sinusoidal, triangular and sawtooth phase detectors when the incoming signal, fading or otherwise, is contaminated with additive white Gaussian noise (AWGN).

14.1 Realization of Various Types of Phase Detectors

a) Sinusoidal PD

In Chapter 7, we have seen that a sinusoidal characteristic of a PD, where the output of the PD is a sinusoidal function of the phase difference $\phi$, is realized by multiplying two sinusoidal waves and taking the lowpass output. A typical configuration of a sinusoidal PD is shown in Fig. 14.1a. Input signal $b$ first band-pass limited and then multiplied with the reference signal after which it is low-pass filtered. Thus if the input signal is $\sqrt{2} a \sin (\omega t + \theta)$ the output of the BP1, is $\frac{4L}{\pi} \sin (\omega t) \cdot 0$. If one takes the reference signal as $\delta \cos \omega t$, then the lowpass output is

$$v_p = \frac{2La}{\pi} \sin \theta$$

(14.1)

where
b) Triangular phase detector

More both the incoming and the reference signals are first limited and then they are multiplied after which it is low-pass filtered (cf. Fig. 14.1b). The waveforms after the limiters are shown in Fig.

\[ \phi = \sin(\phi) - \cos(\phi) + 9 \]  

(14.2)

Fig. 14.1. The sinusoidal and triangular phase detector models.

14.2a. The corresponding multiplied output is also shown in Fig. 14.2b. Therefore, the low-pass output is given by

\[ e^L \]

Fig. 14.2. Illustrating operation of a triangular phase detector.
The phase lock theories and applications

\[ v(\phi) = \frac{j\phi}{\pi} \int_0^{\phi} v(\theta) \, d\theta \]

\[ = \frac{2L}{\pi} \left[ \int_0^{\phi} \frac{d\theta}{\pi} - \int_{\pi/2}^{\phi + \pi/2} \frac{d\theta}{\pi} \right] \]

\[ = \frac{4L}{\pi} \phi - \frac{2L}{\pi} \phi - \pi/2 < \phi < \pi/2 \quad (14.3) \]

Similarly,

\[ v(\phi) = \frac{2L}{\pi} \left( \frac{\pi}{2} - \phi \right) \quad \pi/2 < \phi < 3\pi/2 \quad (14.4) \]

Again the two limiting outputs can be written as

\[ v(\phi) = \frac{4L}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \sin(2m+1)(\omega t + \theta) \]

and

\[ v(t) = \frac{4L}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \cos(2m+1)(\omega t + \theta) \]

Therefore, the average output is

\[ \bar{v}(\phi) = \frac{4L}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \sin(2m+1)\phi \quad (14.5) \]

which is the Fourier series expansion of the periodic triangular waveform, as represented by (14.3) and (14.4).

\( c \) Sawtooth Phase Detector

A typical block diagram of a sawtooth PD is shown in Fig. 14.3a.

\[ \text{Fig. 14.3. The sawtooth phase detector model.} \]

The outputs at the different points of Fig. 14.3a are shown in Fig. 14.4. Looking at the output one finds that the output of the integrator is
Fig. 11.4. Illustrating operation of a sawtooth phase-detector.

\[ v(t) = \frac{I_a}{f} \left( \int_{-\pi}^{t} v(t')dt' - \int_{0}^{t} \int_{0}^{t'} v(t'')dt'' \right) \]

\[ = \frac{I_a}{f} \left( \Delta t - T \right) = \frac{I_a}{f} \left( \phi - \pi \right) \quad 0 < \phi < 2\pi \]

By suitably referencing, i.e., adjusting initial phase difference, the above result can be written as

\[ v(t) = \frac{I_a}{\pi} \phi \quad -\pi < \phi < \pi \]

This gives the sawtooth characteristics.

The above characteristic can be realized with the help of the
equivalent block diagram of Fig. 14.5. The limiter outputs of the two channels are respectively given by

\[ v_1(t) = \frac{4L}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin \left( \frac{(2m+1) \pi t}{T} \right) \]

\[ v_2(t) = \frac{4L}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin \left( \frac{(2m+1) \pi t}{T} \right) \sin \left( \frac{2m+1}{2m+1} \right) \]

where the limiter height is taken to be unity for the sake of simplicity.

Therefore, the low-passed output of the product of \( v_1(t) \) and \( v_2(t) \) is

\[ i(t) = -8 \sum_{m=0}^{\infty} \frac{1}{2^{2m+1}} \cos \left( \frac{2m+1}{2m+1} \right) \]

After carrying the operation as indicated in Fig. 14.5, one gets

Fig. 14.6a. The additive noise response characteristic of a sinusoidal phase detector.
The mathematical model of a sawtooth phase detector.

\[
\nu_1(t) = \frac{4L}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin[(2m + 1) (\omega_0 t + \varphi)]
\]

and

\[
\nu_2(t) = \frac{4L}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin[(2m + 1) \omega_0 t]
\]

where the limiter height is taken to be unity for the sake of simplicity.

Therefore, the low-passed output of the product of \(r_1(t)\) and \(\nu_2(t)\) is

\[
\eta(t) = -\frac{8}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m + 1)^2} \cos (2m + 1) \varphi
\]

(14.6)

After carrying out the operation as indicated in Fig. 14.5, one gets

![Graph showing the additive noise response characteristic of a sinusoidal phase detector.](image-url)
The additive noise response characteristics of a triangular phase detector.

\[ \nu(\phi) = \frac{16}{\pi^2} \left( \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \cos(2m+1) \phi \right) \times \left( \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \sin(2m+1) \phi \right) \]  

(14.7)
which is the Fourier series expansion of a sawtooth wave of maximum height unity. With these few words let us analyze the behavior of these three types of phase detectors to noisy signals. Before we do, let us first characterize the net input signal to the phase detector.

14.2 Signal Plus AWGN to PD Input

We assume that the phase detector is preceded by a narrow band filter. Therefore, the input signal is written as

\[ r(t) = 2A \sin (\omega_0 t + \theta) + \sqrt{2n_0(t)} \sin (\omega_0 t + \theta) \]

and

\[ r(t) = \sqrt{2n(t)} \sin (\omega_0 t + \theta + \Psi(t)) \]  \hspace{1cm} (14.8)

where

\[ a(t) = [(A - n_0(t))^2 + n_0^2(t)]^{1/2} \]

and

\[ \tan \Psi(t) = \frac{n_0(t)}{A - n_0(t)} \]  \hspace{1cm} (14.9)

Putting

\[ x(t) = (A - n_0(t)) = a(t) \cos \Psi \]

and

\[ y(t) = (n_0(t)) = a(t) \sin \Psi \]  \hspace{1cm} (14.10)

and remembering that \( \sqrt{2n_0(t)} \) and \( \sqrt{2n(t)} \) are independent Gaussian noise with zero mean and variances \( \sigma^2 \), we can write down the joint probability density function of \( x(t) \) and \( y(t) \) as

\[ p(x, y) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left( - \frac{(x^2 + y^2)}{2\sigma^2} \right) \times \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left( - \frac{y^2}{2\sigma^2} \right) \]  \hspace{1cm} (14.11)

\[ = \frac{1}{\sigma^2} \exp \left( - \frac{x^2 + y^2 + 2y^2}{2\sigma^2} \right) \]  \hspace{1cm} (14.12)

From (14.10), one finds that

\[ x^2 + y^2 = a^2(t) \]

and therefore, (14.12) can be written as

\[ p(x, y|\Psi, \sigma) = \frac{1}{\sigma^2} \exp \left[ - (a^2 + \Psi^2 - 2a \cos \Psi)\sigma^2 \right] \]
Therefore,
\[ p(\alpha, \psi) = \frac{d}{dp} \exp \left[ - \left( \psi^2 + A^2 - 2A \cos \psi \gamma \beta \right) \right] \]  
(14.13)

[use \( p(\gamma, \psi, \delta \psi, \alpha) \delta \psi = p(\gamma, \psi, \alpha) \delta \psi \) and \( d\gamma d\beta = d\gamma d\beta \)]

Now it is to be noted that in this chapter we are interested to find the signal gain of a phase detector. We will now be calculating the output signal process after the limiting. We will here follow Linsley's [4] method. He has shown that the output signal process is defined as
\[ S_d(t) = E[r(t)/S_0(t)] \]  
(14.14)
where \( S_d(t) \) is the output signal component and \( S_0(t) \) is the input signal to the limiter. Moreover, \( r(t) \) denotes the total output of the limiter.

Now assuming the limiter characteristic as
Limiter Output = Sgn [Input].

Therefore, the conditional expectation in (14.14) can be evaluated as
\[ S_d(t) = \int_0^{\infty} \text{Sgn} \left[ S_0(t) + \eta(t) \right] \rho(n) \, dn \]  
(14.15)
where \( \rho(n) \) is the pdf of the noise. Here we have used
\[ p(r(t)/S_0(t)) \approx p(r(t) - S_0(t)) \]  
(14.16)

Therefore, using (14.15) one can write
\[ S_d(t) = \int_0^{\infty} \exp \left( -\frac{\eta(t)^2}{2\sigma^2} \right) \, dn \]
(14.17)

Further, the output of the limiter can be expanded in the Fourier series as
\[ S_d(t) = \sum_{k=1}^{\infty} \mu_k \sin k (\omega_0 t + \phi) \]  
(14.18)
where

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} E \eta_\delta(t) \delta(t') \sin k \nu d \nu \]  
(14.19)

where \( \nu = \alpha_d + \theta \)  
(14.20)

Thus we write \( b_n \) as (cf. 14.17)

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin \nu}{\sqrt{2\pi}} \sin \nu \exp (-x^2/2) \sin k \nu d \nu dY \]  
(14.21)

Putting \( p = k \nu \)

\[ \frac{\pi}{2} b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin k \nu d \nu dY \int_0^\infty \exp (-x^2/2) dx \]

Putting \( x = \sqrt{2p} \sin \theta \)

one gets

\[ \frac{\pi}{2} b_n = \frac{\sqrt{2p}}{\pi} \int_{-\pi}^{\pi} \sin k \nu d \nu dY \int_0^\infty \exp (-p \sin^2 \theta) \cos \theta d \theta \]

\[ = \frac{\sqrt{2p}}{\pi} \int_{-\pi}^{\pi} \sin k \nu d \nu dY \int_0^\infty \exp (p/2) \exp \left( \frac{\theta \cos \theta}{2} \right) \cos \theta d \theta \]  
(14.22)

Using

\[ \exp \left( \frac{\theta \cos \theta}{2} \right) = \sum_{n=0}^\infty \frac{\cos (\theta/2)^n}{n!} \cos 2n \theta \]

The above integral can be evaluated easily and the result is

\[ b_n = b_{n+1} - \frac{4}{\pi (2m+1)} f_\alpha(\phi) \]  
(14.23)

where

\[ f_\alpha(\phi) = \frac{\sqrt{2p}}{\pi} \exp (-p/2 \left[ \psi_0(\phi/2) + I_{\text{ext}}(\phi/2) \right]) \]  
(14.24)
14.3 Noise Analysis of Phase Detectors

The transfer characteristics of the phase detector is represented by $f(\phi)$, which is an odd periodic function of $\phi$. This is normalized with respect to its maximum value of unity and bounded within modulo-2$\pi$. Thus $f(\phi)$ can be represented by

$$f(\phi) = \sum_{n} C_n \sin n\phi$$

(14.25)

where the coefficient $C_n$ determines the type of phase detector. For phase detectors of practical interest, namely, sinusoidal, triangular and sawtooth varieties, $C_n$'s can be obtained from the Fourier series expansion of the transfer characteristics [2] and are shown in Table 1.

<table>
<thead>
<tr>
<th>Sinusoidal</th>
<th>Triangular</th>
<th>Sawtooth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_0 = 1$</td>
<td>$C_{0m+1} = (-1)^m \frac{2}{\pi \sqrt{2m+1}}$</td>
<td>$C_{2m+1} = \frac{2}{\pi} \frac{1}{2m+1}$</td>
</tr>
<tr>
<td>and</td>
<td>$C_n = 0$, for $n = 2m + 1$</td>
<td>$C_{2m} = \frac{(-1)^{m+1}}{2m}$</td>
</tr>
<tr>
<td>$C_0 = 0$, for $n = 2m$</td>
<td>$C_n = 0$, for $n = 2m$</td>
<td>$C_{2m} = \frac{(-1)^{m+1}}{2m}$</td>
</tr>
<tr>
<td>3 etc.</td>
<td>3 etc.</td>
<td>3 etc.</td>
</tr>
</tbody>
</table>

Now referring to (14.8) one finds that the net input signal to the phase detector is $\sqrt{2\alpha(t)} \sin (\omega t + \theta + \Psi(t))$. Thus the phase detector output in this case can be written as

$$v(\phi) = f(\phi + \Psi) = \sum C_n \sin n(\phi + \Psi)$$

(14.26)
\[ \varphi + \Psi = (\varphi_1 - \varphi_2) f + \Phi_0 + \Psi \]  
(14.27)

Here \( \varphi \) denotes the phase difference between the signal component and the reference signal.

Hence it is observed that in the presence of AWGN, the instantaneous output of the phase detector becomes a function of the instantaneous phase of the signal-plus-noise relative to the phase of the reference signal. The noise phase process is comparatively faster than the relative signal phase process \( \varphi(t) \). As such the PD output will receive a large number of noise samples within a period when \( \varphi \) has not changed appreciably. Thus the average phase detector output will be given as the average of (14.26) taken over all \( \Psi \), i.e.,

\[ \langle \varphi(t) \rangle = \int_0^{2\pi} f(\varphi + \Psi) \rho(\varphi, \Psi) \, d\Psi \]

(14.28)

Then changing the order of integration i.e., first over \( \Psi \) and then over \( \varphi \), one obtains [3]

\[ v(t) = \sum_{m=0}^\infty C_{m+1} \sqrt{2m} \exp \left( -\frac{m}{2} \right) \frac{\Gamma(m/2)}{\Gamma(m + 1)} \varphi + \sum_{m=2}^\infty C_{m} \exp \left( -\frac{m}{2} (m-1)! \right) \left( \frac{m}{2m} \right)^m 
\]

\[ \frac{1}{2m} \left[ l(\varphi(m + 1; 2m; \rho) - l(\varphi(m + 2; 2m + 2; \rho) \right \} \sin 2m \varphi \]

(14.29)

where the first part of the right hand side is for odd \( m \) and second part for even \( m \). Here \( p = A \) is the carrier-to-noise power input at the PD input, \( I_0(X) \) is the modified Bessel function of argument \( X \) and order \( m \) and \( F_1(\alpha; b; z) \) is the degenerate Hypergeometric function of argument \( z \) and parameters \( \alpha \) and \( b \) defined as.
Phase Detector Response to Nyquist Signals

\[ F(a; b; z) = 1 + \frac{a}{b} z + \frac{a(a + 1)}{b(b + 1)} z^2 \]  

(14.30)

\[ e^{-i} F(a; b; z) = F(b - a, b; -z) \]  

(14.31)

\[ \frac{d}{dz} F(a; b; z) = \frac{a}{b} F(a + 1, b + 1; z) \]  

(14.32)

If \( m \) is a half integer and \( n \) is an integer, \( F(a; b; z) \) can be expressed in terms of \( \Gamma(t) \) and \( \cos (\varphi/2) \). Thus

\[ F(\varphi; 1; -z) = \exp (-z/2) \Gamma(z/2) \]  

and

\[ \exp (-z/2) \Gamma(z/2) \]  

Although the equation (14.32) gives the general expression for the phase detector output having either a sinusoidal, triangular or sawtooth characteristic, in the following we derive the results, based on the model of the phase detectors (5, 6).

a) Sinusoidal PD

Putting \( m = 0 \), in (14.23), (14.24) and (14.11) we write the first zonal signal output of the BPL as (Fig. 14.14).

\[ S(t) = \frac{d}{dt} F(\varphi, \varphi) \sin (ae t + \theta) \]  

(14.33)

where

\[ F(\varphi) = \sqrt{\frac{\pi}{2}} \exp (-\varphi^2) \Gamma(\varphi/2) + I(\varphi/2) \]  

(14.34)

Therefore, the output of the PD is obtained by multiplying \( S(t) \) with the reference input, \( 2 \cos (\omega t) \) and taking the low- and high-level output. This is given by

\[ \sin \theta \]  

(14.35)

where the output has been normalized to unity with respect to the maximum value.
b) Triangular PD

We have already seen that here both the input and the reference signals are hard limited before being multiplied and low pass filtered (Fig. 14.1b). Thus the output of the limiters are (14.18, 14.23 and 14.24).

\[ S_d(t) = \frac{4}{\pi} \sum_{n=\pm \infty} \frac{f_d(n)}{2n+1} \sin (2n+1) (\omega t + \phi) \]  

(14.36)

and the limiter output for the reference signal is

\[ S_r(t) = \frac{4}{\pi} \sum_{n=\pm \infty} (-1)^n \cos (2n+1) (\omega t) \]  

(14.37)

Therefore, carrying out the operations as indicated in Fig. 14.1a one finds

\[ v_d(t) = \frac{8}{\pi^2} \sum_{n=\pm \infty} \frac{f_d(n)}{(2n+1)^2} \sin (2n+1) \sin (\omega t) \]  

(14.38)

where

\[ f_d(n) = \sqrt{\frac{2}{\pi n}} \exp (-n/2) \left[ I_n (n/2) + I_{-n} (n/2) \right] \]

(14.39)

c) Sawtooth PD

Referring to the mathematical model of the sawtooth phase detector of Fig. 14.5, we write the following expressions for the outputs at different points of Fig. 14.5. The limiter output for the reference signal is

\[ S_d(t) = \frac{4}{\pi} \sum_{n=\pm \infty} \frac{1}{2n+1} \sin (2n+1) (\omega t) \]  

(14.39)

The low pass output of the product of \(S_d(t)\) and \(S_r(t)\) will be

\[ v_d(t) = \frac{8}{\pi^2} \sum_{n=\pm \infty} \frac{f_d(n)}{(2n+1)^2} \cos (2n+1) \omega t \]  

(14.40)

We now add unity to \(v_d(t)\) and then multiply the resultant by \(\Phi / \omega q\) to get the sawtooth wave. Thus the output of the sawtooth phase detector is given by

\[ v_d(t) = \frac{16}{\pi^2} \left[ \frac{1}{\omega t} - \frac{2}{\pi} \sin (2n+1) \cos (2n+1) \Phi t \right] \]  

(14.41)
The variations of the outputs of the three types of phase detectors with \( \varphi \) for different values of the input carrier-to-noise ratio are shown in Fig. 14.6. From the Fig. 14.6 one finds that the outputs in all the three cases become sinusoidal for low values of the input CNR. This is also readily seen by referring to (14.33), (14.38) and (14.41). These are given by

\[
V(\varphi) \left|_{\text{low}} \right. = \frac{1}{n} \sqrt{\frac{2}{\pi}} \sin \varphi
\]

(14.42a)

\[
V(\varphi) \left|_{\text{hi}} \right. = \frac{3}{2} \frac{1}{n} \sqrt{\frac{2}{\pi}} \sin \varphi
\]

(14.42b)

and

\[
V(\varphi) \left|_{\text{low}} \right. = \frac{2}{n} \sqrt{\frac{2}{\pi}} \sin \varphi
\]

(14.42c)

14.4 Remarks on the Phase Detector Gain and Threshold Properties

Remembering that the maximum value of the phase detector output in the above three cases has been taken to be unity, we now plot the variation of the signal gain of the phase detector, defined as

\[
G = \left[ \frac{V(\varphi)}{V(\varphi)} \right]_{\varphi = 0}
\]

(14.43)

with input CNR. This is shown in Fig. 14.7. Now if we define the threshold CNR for the phase detector as that value of the CNR at which the signal gain falls to 0.707 times the maximum gain, then one gets the following values of threshold CNRs:

\[
\text{CNRT}_{\text{low}} = 0.177 \text{ dB}
\]

(14.44)

\[
\text{CNRT}_{\text{hi}} = -2.396 \text{ dB}
\]

(14.45)

and

\[
\text{CNRT}_{\text{low}} = -3.47 \text{ dB}
\]

(14.46)

The corresponding gains of the phase detectors at threshold CNR are then given by

\[
(G)_{\text{low}} = 0.707
\]

(14.47)

\[
(G)_{\text{hi}} = 0.45
\]

(14.48)

and

\[
(G)_{\text{low}} = 0.225
\]

(14.49)
Scanning the expressions (14.45) through (14.48), one concludes that the sawtooth phase detector has the lowest threshold CNR although it has the lowest signal gain. On the other hand, the sine type phase detector has the opposite character. Moreover, in making a choice of the type of phase detectors, one must take into consideration the rate of variation of the signal gain around the threshold point. These are respectively 0.195, 0.233 and 0.133 for the sinusoidal, triangular and sawtooth phase detectors. However, if one considers the transfer characteristics of the PD's, having identical slopes at $\varphi = 0$, the corresponding values of the gain variation at threshold CNR's are 0.193, 0.360 and 0.418. Finally, the choice of a particular type of phase detector for a phase locked loop should not be based solely on the basis of the noise performance, one has to think of the other requirements such as pull-in characteristics, tracking behaviour and the noise bandwidth of the loop.

14.5 Response of a PD to Noisy Fading Signals

Normally the signal, before arriving at the receiver, travels through time varying channels like the Rayleigh, Rician and log-normal,
which represent a wide class of time varying channels based upon the interaction of a radio wave and turbulent medium (7). As a result the received signal in the presence of AWGN \( (a(t)) \) may be represented as

\[
r(t) = \sqrt{2A} \sin \psi(t) + a(t) \tag{14.50}
\]

where

\[
\psi(t) = \omega t + \theta(t)
\]

The amplitude, \( a(t) \), and the phase \( \theta(t) \) are random processes and depend on the types of channels.

For example, if the transmitted signal passes through a Kolmogorov channel, the received signal in the absence of the AWGN can be written as

\[
r_d(t) = \sqrt{2X_d(t)} \sin (\omega_d t + \theta_d) + \sqrt{2X_d(t)} \cos (\omega_d t + \theta_d) - \sqrt{2X_d(t)} \sin (\omega_d t + \theta_d) \tag{14.51}
\]

where \( X_d(t) \) and \( X_d(t) \) are narrow-band, independent, zero mean, stationary, Gaussian random processes with variance \( \delta^2/2 \).

Therefore,

\[
r_d(t) = \sqrt{2a(t)} \sin (\omega_d t + \theta(t)) + \delta(t) \tag{14.52}
\]

Referring to (14.13) we find that the joint pdf of \( a \) and \( \psi \) is given by

\[
p(a, \psi) = \frac{a}{\pi a^2} \exp \left( -\frac{(a \psi + \psi_A - 2A \cos \theta)\psi}{a^2} \right) \tag{14.53}
\]

and

\[
\tan \theta = \frac{\chi(t)}{A - \chi(t)} \tag{14.54}
\]

Therefore,

\[
p(\theta) = \int_0^{2\pi} p(a, \psi) d\psi \tag{14.55}
\]

Putting

\[
\exp \left( \frac{2aA \cos \theta}{\delta^2} \right) = \sum_{m=-\infty}^{\infty} \exp \left( \frac{2mA \cos \theta}{\delta^2} \right) \cos m\psi
\]

\[
\exp \left( \frac{2A \cos \theta}{\delta^2} \right) = \sum_{m=-\infty}^{\infty} \exp \left( \frac{2mA \cos \theta}{\delta^2} \right) \cos m\psi
\]
one gets
\[
p(\alpha) = \frac{2a}{b_0^2} \exp \left( -\frac{a^2 + A^2}{2b_0^2} \right) L \left( \frac{2aA}{b_0^2} \right) \quad \text{for} \quad \alpha > 0
\]
\[= 0 \quad \text{otherwise}
\]  
(14.57)

Similarly
\[
p(\alpha) = \int \rho(\alpha, 0) \, d\alpha
\]

The integration requires a number of steps. However, the result is
\[
p(\alpha) = \frac{1}{2\pi} \exp \left[ 1 + \sqrt{\gamma} \cos \theta \exp (\gamma \cos^2 \theta) \right] \times \text{erf} \left( \frac{\gamma \cos \theta}{\sqrt{\gamma}} \right)
\]  
(14.58)

where
\[
\gamma = \frac{A^2}{b_0^2}
\]

and
\[
\text{erf} \, (x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} \exp (-t^2/2) \, dt
\]  
(14.59)

For very small \( \gamma \) \((\ll 0.1)\), \( p(\theta) \) can be written as
\[
p(\theta) = \frac{1}{2\pi} \left( 1 + \sqrt{\gamma} \cos \theta \right)
\]  
(14.60)

Note that \( \gamma \) denotes the ratio of the specular power to the fading power \( b_0^2 \). Further if the transmitted signal is denoted by \( A_0 \), \sin (\alpha x + \theta) \) then the ratio of the total power at the output of the channel, \( A_x + B_x \), to the power into the channel, \( A^2 \), lies between zero and one and it is denoted by \( \alpha \). Thus
\[
\alpha = \frac{A^2 + B^2}{A^2}
\]  
(14.61)

i.e.,
\[
\alpha \beta^2 = A^2 (1 + \gamma)
\]  
(14.62)

Incidentally, the distribution function \( p(\alpha) \) is the Ricean density function. Hence the channel is known as the Ricean channel. For \( \gamma \ll 1 \), the result of (14.57) can be written as
\[
p(\alpha) = \frac{2a}{b_0^2} \exp \left( -\frac{a^2}{b_0^2} \right), \quad \text{for} \quad \alpha > 0
\]
\[= 0 \quad \text{otherwise}
\]  
(14.63)
which is the Rayleigh density function and \( p(\theta) \) is given by

\[
p(\theta) = \frac{1}{2\pi} \exp\left(-\frac{(\theta - \theta_0)^2}{2\Delta^2}\right) \quad (14.64)
\]

for \( \theta < \theta_0 < 2\pi \), otherwise \( (14.65) \)

Now when a signal enters into a lognormal channel with an amplitude \( A_0 \), it exists with an amplitude \( a(t) \) given by

\[
a(t) = A_0 \exp(M(t))
\]

where \( M(t) \) is a Gaussian random variable with the mean \( \mu_m \) and variance \( \sigma_m^2 \). The first order probability density function of \( a(t) \) is given by

\[
p\left(\frac{a}{A_0}\right) = \frac{a}{A_0} \cdot \frac{1}{\sqrt{2\pi}\sigma_m^2} \exp\left(-\frac{(\ln(a/A_0) - \mu_m)^2}{2\sigma_m^2}\right) \quad (14.67)
\]

using conservation of power principle, it can be shown \( [7] \) that

\[
\mu_a = -\frac{\sigma_m^2}{2} \quad (14.68)
\]

The phase fluctuation in this case is not precisely known, however, in some cases the phase fluctuation is a Gaussian random process \( \delta(t) \). With these few words about the nature of signals appearing at the outputs of Rayleigh, Rician and lognormal channels, let us now turn to the study of the phase detector response to such signals when accompanied by AWGN.

Now it is important to note that the bandwidth of the fading process is much small compared to that of AWGN. This indicates that before \( a(t) \) and \( \delta(t) \) change significantly, the phase detector receives a fairly large number of samples of AWGN. Therefore, we can first compute the signal gain of the phase detector by assuming \( a(t) \) and \( \delta(t) \) to be nearly constant. We follow the methods of section 14.5. Thus in this case the phase detector output can be written as

\[
v\left(\theta, a, \delta\right) = \mathcal{A}\left(\theta, a, \delta\right) \quad (14.69)
\]

once this is done the average signal gain of the PD is given by

\[
v(\theta) = \int_{-\infty}^{\infty} f(\theta, a, \delta) p(a, \delta) \, da \quad (14.70)
\]
Note that the nature of $f(t)$ depends on the type of PD. This way of tackling the problem is rather difficult, and in order to simplify our discussion we will consider the following cases commensurate with practical situations [7].

Case I: Slow Fading Rayleigh and Rayleigh Channels
In this case the variation of $\theta(t)$ is so slow that it can be treated as a constant compared to the $a(t)$ process. Thus the signal gain of the phase detector in this case becomes

$$v(t) = \int_0^t f(\theta | a) a(t) dt$$  \hspace{1cm} (14.71)

where $f(\theta | a)$ s for the sinusoidal, triangular and sawtooth types of PD are respectively given by (14.35), (14.38) and (14.41). The $\theta$ of the above equations should now be taken as

$$\theta = \frac{\partial f}{\partial a}$$  \hspace{1cm} (14.72)

The computation of (14.71) should be done through a digital computer and results of computation are shown in Fig. 14.8.

![Graph showing the response characteristic of a sinusoidal phase detector to a slowly fading noisy signal.](image)

Fig. 14.8a. Response characteristic of a sinusoidal phase detector to a slowly fading noisy signal.
Fig. 14.8a. Response characteristics of a triangular phase detector to a slowly fading noisy signal.

Fig. 14.8b. Response characteristics of a sawtooth phase detector to a slowly fading noisy signal.

Case II: Weak Amplitude Fading
In such a case the instantaneous received amplitude $a(t)$ remains almost constant, i.e., $a(t) = A$, i.e., $A^2 = \sigma_f^2$. Therefore $p(b)$ is obtained from (14.58) by putting large values of $\sigma^2$ and is given by
\[ p(\theta) = \frac{\gamma}{\sqrt{\pi}} \exp\left(-\gamma^2 \theta^2\right) \]  
(14.73)

Hence the phase detector output is given by

\[ \langle \phi(\theta) \rangle = \frac{\gamma}{\sqrt{\pi}} \int_0^\infty \phi(\theta) p(\theta) d\theta \]
(14.74)

Now using (14.35), (14.38) and (14.41) and replacing \( \phi \) by \( \phi + \epsilon \) as to obtain \( \phi(\theta) \) and from (14.74) we get the following expressions for the signal gains of the sinusoidal, triangular and sawtooth phase comparators

\[ \text{Sinusoidal:} \quad \phi(\theta) = \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{f_m(\theta)}{(2m + 1)^2} \sin (2m + 1)\theta \]
(14.75)

\[ \text{Triangular:} \quad \phi(\theta) = \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{f_m(\theta)}{(2m + 1)^2} \sin (2m + 1)\theta \]
(14.76)

\[ \text{Sawtooth:} \quad \phi(\theta) = \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{f_m(\theta)}{(2m + 1)^2} \sin (2m + 1)\theta \]
(14.77)

The transfer characteristics of the PD's are plotted in Fig. 14.9.

The expressions for the signal gains of the phase detector as given by (14.71) through (14.77), can be used to calculate the signal gains of the PD's for the Rayleigh channel by putting \( \gamma = 3 \). The results for the lognormal channel can also be deduced with the help of (14.71) and (14.75) by using the pdf's \( p(\theta) \) and \( p(\theta) \) for the lognormal channel. \( p(\theta) \) is given by (14.67) whereas \( p(\theta) \) is assumed to be given by
fig. 14.9a. Sinusoidal phase detector response to a noisy weak amplitude fading signal.

fig. 14.9b. Triangular phase detector's response to a noisy weak amplitude fading signal.

\[ \rho(\theta) = \frac{1}{\sqrt{2\pi b^2}} \exp \left( -\frac{\theta^2}{2b^2} \right) \]  \hspace{1cm} (14.78)

where \( b \) is the variance of the \( b \)-process.

Referring to all the figures representing the signal gains of the PD's for fading signals, it is observed that the gains of PD's are reduced considerably. This indicates acquisition or tracking difficul-
ties. From the comparative study of the PD-characteristics, we further find that the deterioration of a particular gain characteristic for a slow fading signal is minimum for a sinusoidal PD and maximum for a sawtooth PD. Also the curves for a sawtooth PD show that the range of linearity of the PD output decreases with fading. The effect of fading with weak amplitude fluctuation is less prominent than that of the slow fading case. A sine type PD is preferable in the condition of deep fading.

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Chapter 15

NONLINEAR ANALYSIS WITH NOISY SIGNALS

One of the outstanding features of a phase-locked loop is that it spontaneously fails in synchronism with the incoming signal, embedded in unwanted disturbances, even if the signal is an angle-modulated one. In such a case, it is not only desirable to evaluate the output signal-to-noise ratio in terms of the input carrier-to-noise ratio but also to estimate the lowest value of the input CNR up to which the output phase variation of the PLL is in union with the input phase variation. The important parameters that give a clue to the solution of these problems are mean square phase error, loop noise bandwidth, probability density function of the phase error, frequency of slipping or skipping cycles, etc.

The essential mathematical tools which are at present available for analyzing the behavior of a phase-locked loop in response to a signal corrupted with a stationary random noise of the Gaussian type are based on either of the following methods, viz., (1) the method of Rice or Fourier series method and (2) Fokker-Planck technique or Diffusion equation method.

Fundamental concept in the method of Rice is the notion of the spectrum of the random noise and the relation between the spectrum and the correlation function. Out of this concept, one proposes the statistical linearization method. The overall advantage of the statistical linearization method is that it admits linear analysis of a nonlinear system. It is worthwhile to mention that quasi-linearization techniques are approximate in the sense that here at the outset one postulates certain statistical properties of the random process at the input to the nonlinear element and then determines the dis-
trinbution densities of the random variables. But in the method of Fokker-Planck or continuous random walk, one starts with the more physically fundamental view point, i.e., by studying the differential equation that determines the distribution densities of the random variables. Unfortunately, the exact solution of the differential equations for higher order nonlinear systems are still not available. In this chapter we will discuss both these methods.

15.1 Random Walk Problems and Markovian Processes

A simple example of a random walk problem may be visualized in this way. Suppose that a man is initially standing in the middle of a wide road; his initial position is denoted by \( x = 0 \), as shown in Fig. 15.1.

![Fig. 15.1. The random walk steps.](image)

He then takes a series of steps of equal length \( L \), each step being taken, either in the direction of left or in the direction of right with equal probability of \( \frac{1}{2} \). Obviously after taking \( N \) steps, the man could find himself at any of the points,

\[-NL, (-N + 1)L, \ldots, -L, 0, L, \ldots, (N-1)L, \text{ and } NL.\]

The question, that now comes up for consideration, is: What is the probability \( P(mL; N) \) that the man arrives at \( mL \) after taking \( N \) steps? To answer this question we proceed as follows.

Before the first step is taken, the probability of going to \( x = L \) or \( x = -L \) is \( \frac{1}{2} \). If we suppose that the man has moved to \( x = L \) after the first step, the man can be either at \( x = 2L \) or \( x = 0 \) when he takes the second step. The probability of ending up at \( x = 2L \) or \( x = 0 \) is \( \frac{1}{2}, \frac{1}{2} \) respectively. However, if the man were at \( x = -L \) after the first step, the second step would have to take either \( x = -2L \) or \( x = 0 \). The probability of ending up at \( x = -2L \) or \( x = 0 \) is again \( \frac{1}{2}, \frac{1}{2} \). Thus the total probability of ending up at \( x = 0 \) is \( 0 + \frac{1}{2} = \frac{1}{2} \).

If we again suppose that the second step has taken the man to \( x = 2L \), the third step can lead him to either \( x = 3L \) or \( x = L \), from \( x = 2L \).

The probability of moving either to \( x = 3L \) or \( x = L \), from \( x = 2L \),
is $1/4 = 1/4$. If the man were at $x = -2L$ after the second step the third step could lead him either to $x = -3L$ or $x = -L$. These are illustrated in Fig. 15.2. Similarly, when the man has taken three steps (cf. Fig. 15.2), the probability $P(L, 3)$ of finding himself at

$$x = L$$ is $3/8$. This can be easily proved. The first step can lead to either $L$ or $-L$. Second step, if started from $L$ can lead to $2L$ or $0$. On the other hand, second step, if begun from $-L$, will lead to $-2L$ or $0$. Thus the third step can start from $-2L$, $0$, $0$ and $2L$. Thus the third step can lead to $L$ in three different ways (starting from $0$, $0$ and $2L$). In computing this result, we have noted that the step taken at $x = mL$, whether moving to the left or to the right, does not depend on the direction of the preceding steps. This sort of property is exhibited by stochastic processes known as Markovian processes. Thus we define a Markov process in the following way, "A stochastic process which has this characteristic, namely that what happens at a given instant of time $t$ depends only on the state of the system at time $t$, is said to be Markoff process" [2]. Let us now compute $P(mL, N)$, i.e., the probability of arriving at $mL$, after $N$ steps. To do this we note that the probability of any given sequence of $N$ steps is $(1/2)^N$. We have already seen that
\[ P(L, 3) = \frac{3}{8} = (\frac{3}{4})^3. \text{ Thus the required probability } P(mL, N) \text{ is} \]
\[ (\frac{3}{4})^m \text{ time number of distinct sequences of steps that will lead to } mL \text{ after } N \text{ steps. We have further seen in order to arrive at } L \text{ after three steps, the man has to take two steps, i.e., } (3 \div 1)/2 \text{ in the direction of right and one step, i.e., } (3 - 1)/2 \text{ in the left direction. It is important to note that if } m \text{ is odd, then } N \text{ has to be odd, and if } m \text{ is even then } N \text{ has to be even. The number of sequence of steps, which takes the man to } mL \text{ after } N \text{ steps is obviously} \]
\[ N!(N + m)! [(N - m)!^2] \]

Hence

\[ P(mL, N) = \left( \frac{3}{4} \right)^m \frac{N!}{(N + m)! (N - m)!^2} \tag{15.1} \]

\[ \text{If } N \text{ is large, it can be shown [2] that:} \]
\[ P(mL, N) = \left( \frac{2}{3} \right)^m \exp \left( -\frac{m}{2} \right) \tag{15.2} \]

The example, we have taken, falls under the head of discrete Markov process. We now consider an example as illustrated in Fig. 15.3. The

![Fig. 153. The noise source exciting a series resistance-inductance circuit.](image)

The voltage source \(v(t)\), having the character of a stationary wideband Gaussian noise of mean zero and variance \(\sigma^2\) excites a series combination of a resistance \(R\) and an inductance \(L\). Thus we write

\[ \frac{dv}{dt} = -\frac{R}{L} v(t) + \frac{1}{L} \xi(t) \tag{15.3} \]

Thus at any time \(t\), the rate of change of the current is dictated by the present value of the current \((I(t))\) and the voltage \(v(t)\). Since \(v(t)\) is a stationary, white Gaussian process, present value of \(v(t)\) is essentially independent of the previous values. Thus the probability
density function of \( \frac{dy}{dt} \) can be specified only by knowing the present value of \( y(t) \). Therefore, \( y(t) \) is identified as a continuous Markov process. Incidentally we mention that a physical process which is described by a first order differential equation with a white Gaussian noise as the forcing function is generally Markovian in character (Chapter 1).

### 15.2 Smoluchowski Equation

Let us consider a Markov process \( y(t) \), starting at \( t = 0 \) with initial value \( y_0 \), varies with the time as shown in Fig. 15.4. Further let us assume that the values of the process at \( t \) and \( t + \Delta t \) are \( x \) and \( y \) respectively.

If \( p(y_j | y_x; t) \) denotes the conditional probability density then the quantity \( p(y_j | y_x; t) dy \) indicates the probability that the value of \( x(t) \) lies in the infinitesimal interval \( y \) and \( y + dy \), given that \( t \) seconds ago its value was \( y_x \). If \( y_0 \) and \( y \) denote respectively the values of the process (Fig. 15.4) at instance \( t \) and \( t + \Delta t \), we can similarly denote the probability density of \( y \) given \( x \) and \( y_0 \) as \( p(y_j | x^*; \Delta t, y_0; \Delta t) \) (Fig. 15.4). Since \( y(t) \) is the output of a Markov process, \( y(t) = y \) depends only on its previous value \( y(t - \Delta t) \) time earlier. That is, for a Markov process, the probability density function,

![Fig. 15.4. Variation of the Markov process with time.](image-url)
conditioned on an arbitrary number of past values taken at arbitrary sampling instants is identically equal to the probability density function conditioned on only the most recent sampled value \( z \).

Hence

\[
p(y \mid z; \Delta t, y_{\Delta t}; t + \Delta t) = p(y \mid z; \Delta t)
\]  

(15.4)

Omitting the time variables for the moment, we can denote the joint probability density of the three samples \( y, z \) and \( y_{\Delta t} \) as \( p(y, z, y_{\Delta t}) \), which by using the laws of probability, can be written as

\[
p(y, z, y_{\Delta t}) = p(z, y_{\Delta t}) p(y \mid z, y_{\Delta t})
\]  

(15.5)

Integrating both sides of (15.5) with respect to \( z \), we get

\[
p(y, y_{\Delta t}) = \int p(z, y_{\Delta t}) p(y \mid z, y_{\Delta t}) \, dz
\]  

(15.6)

i.e.,

\[
p(y, y_{\Delta t}) = \int p(z, y_{\Delta t}) p(y \mid z) \, dz
\]  

(15.7)

Again writing

\[
p(y, y_{\Delta t}) = p(y \mid y_{\Delta t}) p(y_{\Delta t})
\]

\[
p(z, y_{\Delta t}) = p(z \mid y_{\Delta t}) p(y_{\Delta t})
\]

and bringing back the time variable, the equation (15.6) can be written as

\[
p(y \mid z; \Delta t + \Delta t) = \int p(z \mid z; \Delta t) p(y \mid z; \Delta t) \, dz
\]  

(15.8)

This is the Smoluchowski equation which is sometimes referred to as Chapman-Kolmogorov equation.

15.3 Fokker-Planck Equation

In order to determine the probability density \( p(y \mid z; t) \), we now use (15.8) along with the initial condition
\[ p(x | y_0; 0) = \delta(x - y_0) \quad (15.9) \]

where \( \delta(x) \) is the Dirac's delta function. In order to arrive at the result, we define the following integral

\[ I = \int R(y) \frac{\partial^2 p(y | x; t)}{\partial y^2} \, dy \quad (15.10) \]

where \( R(y) \) is an arbitrary analytic function of \( y \), selected so that

\[ \frac{\partial^2 p(y)}{\partial y^2} \to 0, \quad \text{as} \; y \to \infty \quad (15.11) \]

Using the fundamental definition of the differential calculus we rewrite (15.10) as

\[ I = \lim_{\Delta t \to 0} \int_{-\infty}^{\infty} R(y) \, dy \left[ p(y | x_0; t + \Delta t) - p(y | x_0; t) \right] \left( \frac{\partial^2 p(y | x; t)}{\partial y^2} \right) \, dy \]

\[ - \int_{-\infty}^{\infty} R(y) \, dy \left( \frac{\partial p(y | x_0; t)}{\partial t} \right) \, dy \quad (15.12) \]

We now use (15.8) to replace \( p(y | x_0; t + \Delta t) \) in (15.12). Thus one gets

\[ I = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_{-\infty}^{\infty} R(y) \, dy \int_{-\infty}^{\infty} p(z | x_0; t) p(z | z; \Delta t) \, dz \right] \frac{\partial^2 p(y | x; t)}{\partial y^2} \, dy \]

\[ - \frac{1}{\Delta t} \int_{-\infty}^{\infty} R(y) \, dy \left( \frac{\partial p(y | x_0; t)}{\partial t} \right) \, dy \quad (15.13) \]

Interchanging order of integration and expanding \( R(y) \) in a Taylor series around \( z \), one gets from (15.13)

\[ I = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_{-\infty}^{\infty} p(z | x_0; t) \sum_{n=1}^{\infty} \frac{R(n)(z)}{n!} \, dz \int_{-\infty}^{\infty} (z - y)^n p(y | z; \Delta t) \, dy \right] \frac{\partial^2 p(y | x; t)}{\partial y^2} \, dy \]

Putting

\[ A_n(c) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} (y - c)^n p(y | z; \Delta t) \, dy, \; n \geq 1 \]

\[ (15.14) \]
one gets
\[ I = \int_{-\infty}^{\infty} R(z) A_d(z) \mathcal{N}(z \mid \gamma_i; t) \, dz \]  \hspace{1cm} (15.15)
putting
\[ u = A_d(z) \mathcal{N}(z \mid \gamma_i; t) \]
\[ du = R(z) \, dz \]
one gets from (15.15)
\[ I = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{d\alpha^n} \int \alpha^n \, d\alpha \]  \hspace{1cm} (15.16)
The choice of \( R(y) \) (cf. 15.1) dictates that
\[ R^{(n+1)}(y) A_d(y) \mathcal{N}(y \mid \gamma_i; t) = 0 \]
After \( n \) successive integrations by parts of (15.10) and subtracting (15.11) from (15.10) one gets
\[ 0 = \int R(y) \, dy \left[ \left( R^{(y)}(y) A_d(y) \mathcal{N}(y \mid \gamma; t) \right) \right] \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{d\alpha^n} \left[ A_d(y) \mathcal{N}(y \mid \gamma_i; t) \right] \]  \hspace{1cm} (15.17)
Since \( R(y) \) is arbitrary, the quantity in braces must be zero. Thus we get
\[ \frac{d^y}{d\alpha} \mathcal{N}(y \mid \gamma_i; t) = \frac{(-1)^n}{n!} \frac{d^n}{d\alpha^n} \left[ A_d(y) \mathcal{N}(y \mid \gamma_i; t) \right] \]  \hspace{1cm} (15.18)
inclusion of the initial condition (15.19) permits us to write (15.18) as
\[ \frac{d^y}{d\alpha} \mathcal{N}(y \mid \gamma_i; t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{d\alpha^n} \left[ A_d(y) \mathcal{N}(y \mid \gamma_i; t) \right] \]  \hspace{1cm} (15.19)
This partial differential equation is known as the Fokker-Planck equation. Referring to (15.14) one finds that \( A_d(z) \) can be written as
\[ A_d(z) = \lim_{\Delta t \to 0} \int_{\gamma_i \Delta t}^{\gamma_i + \Delta t} \mathcal{N}(x \mid \gamma; \Delta t) \, dx \]  \hspace{1cm} (15.20)
where

\[ A_x(t) = \lim_{\Delta t \to 0} \frac{\Delta \phi(t)}{\Delta t} \]  \hspace{1cm} (15.206)

13.4 Noise Performance of a First Order PLL

Referring to the block diagram of a first order PLL with the input as

\[ v_i(t) = \sqrt{2} A \sin (\omega_d t + \theta) + n(t) \]  \hspace{1cm} (15.21)

where

\[ n(t) = \sqrt{2} n_0(t) \cos \omega_d t - \sqrt{2} n_0 \sin \omega_d t \]  \hspace{1cm} (15.22)

Properties of \( n_i(t) \) and \( n_0(t) \) are defined in Chapter 12. Now writing the output of the VCO as

\[ v_o(t) = \sqrt{2} V_o \cos (\omega_d t + \phi_0(t)) \]

the output of the phase detector is written as (cf. 9.25)

\[ v_p = K P \left( A \sin \phi + N_i(t) \right) \]  \hspace{1cm} (15.23)

Therefore, the loop equation for a first order PLL (cf. 9.32, \( b_c = 0 \)) is

\[ \frac{d\phi}{dt} = -K \left( A \sin \phi + N_i(t) \right) \]  \hspace{1cm} (15.24)

where \( N \) is a Gaussian noise with one sided spectral density \( N_v \).

Let us now find the probability density function \( p(\phi; t) \). To do this we first assume that \( N(\phi, t) \) white Gaussian noise. This assumption is fairly correct if the bandwidth of the loop is small compared to the bandwidth of the filter preceding the loop. Thus referring to (15.24) one finds that \( p(\phi; t) \) is a Markov process, because \( \frac{d\phi}{dt} \) does not depend on the past history of the process \( \phi(t) \) and it just depends on the present value of \( \phi(t) \). Thus \( p(\phi, t) \) satisfies the equation

\[ \frac{\partial}{\partial t} p(\phi; t) = -\frac{0}{2K} [A(\phi) p(\phi; t)] + \frac{1}{2K^2} [A(\phi) + (\phi; t)] \]  \hspace{1cm} (15.25)

since

\[ A_i(\phi) = 0, \quad n > 3 \]  \hspace{1cm} (15.26)
This has been shown by Yiterbi [9]. For a first order differential equation process, $A_d(\phi)$ vanishes for $n > 2$.

To find the values of $A_d(\phi)$ and $A_d(\delta)$, we integrate both sides of (15.26) over an infinitesimal interval $i$ to $i + \Delta i$. Thus

$$\Delta \phi = \Omega \cdot \Delta t - AK \sin \phi(t) \cdot \Delta t - K \int_{\phi(t)}^{\phi(t + \Delta t)} N(\theta) d\theta \quad (15.27)$$

Hence (cf. 15.20a)

$$A_d(\phi) = \lim_{\Delta t \to 0} \frac{E(\Delta \phi \mid \phi)}{\Delta t}$$

i.e.,

$$A_d(\phi) = \Omega - AK \sin \phi$$

(15.28)

since $N(\theta)$ is a white Gaussian noise with zero mean.

$$A_d(\phi) = \lim_{\Delta t \to 0} \frac{E(\Delta \phi^2 \mid \phi)}{\Delta t}$$

$$= \frac{\lim_{\Delta t \to 0} K^2}{\Delta t} \int_{\phi(t)}^{\phi(t + \Delta t)} \int_{\phi(t)}^{\phi(t + \Delta t)} E[N(\theta) N(\phi)] d\theta d\phi$$

$$= \frac{\lim_{\Delta t \to 0} K^2}{\Delta t} \int_{\phi(t)}^{\phi(t + \Delta t)} N_0 \frac{\theta - \phi(t)}{2} d\theta$$

i.e.,

$$A_d(\phi) = \frac{KN_0}{2} \frac{\phi(t) - \phi}{\Delta t}$$

(15.29)

Inserting the coefficients $A_d(\phi)$ and $A_d(\phi)$ in the Fokker-Planck equation one gets

$$\frac{\partial P(\phi, t)}{\partial t} = \frac{\partial}{\partial \phi} \left( (\Omega - AK \sin \phi) P(\phi, t) \right) + \frac{KN_0}{4} \frac{\partial^2 P(\phi, t)}{\partial \phi^2}$$

(15.30)

Before attempting to solve for $P(\phi, t)$, let us first look into the nature of this distribution physically. Let us further simplify the situation by assuming that the open loop frequency error is zero. Thus before the noise source is injected, the probability density function of the phase error at time $t = 0$ is given by

$$P(\phi \mid 0) = \delta(\phi)$$

(15.31)
This is shown in Fig. 15.5. As time passes, the noise will produce

\[ f(t) \]

\[ P(t) \]

\[ P(t_1) \]

\[ P(t_2) \]

\[ P(t_3) \]

Fig. 15.5. Qualitative illustration of the phase error probability density function of a PLL with the passage of time

\[ f \rightarrow t \rightarrow t_1 \rightarrow t_2 \rightarrow t_3 \]

phase jitter. Thus after a time \( t - t_0 \), the probability density function will look like that of Fig. 15.5b. But as time increases there will be moments when the PLL will be thrown out of lock momentarily. After a very short time the loop will again achieve lock. In doing so the loop may lose or gain a cycle and the loop will be operating either around \( 2\pi \) or \(-2\pi\). Thus the pdf of \( \theta \) will appear around \(-2\pi\) or \(2\pi\) as illustrated in Fig. 15.5 (c). Thus after a sufficiently long period, the pdf will appear as a multimodal function, each centred around the multiple of \( 2\pi \). The central mode, centred around 0, will, however, be maximum. Thus in the long run the pdf will have a steady pattern and it will be spread over all \( \theta \). Again referring to the Fokker-Planck equation, one finds that it is a nonlinear equation with a periodic coefficient in \( \theta \). Thus if \( P(\theta; t) \) is a solution for the initial condition \( \theta = \theta_0 \), then \( P(\theta + 2\pi; t) \) is also a solution.
for the initial condition \( \gamma(0, t) = \varphi_0 + 2\pi n \), where \( n \) is any integer. Thus to get rid of this ambiguity of \( 2\pi \), we define a new solution \( \phi(y, t) \), such that
\[
\phi(y, t) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \left[ (\Omega - AK) \sin \gamma \right] u(\gamma; t)
\]
\[+ \frac{K^2 T}{2} \int_0^\infty B(\gamma; t) \frac{d\gamma}{2\pi}, \tag{15.33}\]
with the initial condition
\[
o(0, t) = \sum_{n=0}^{\infty} \delta(\gamma - q_0 - 2\pi n), \tag{15.34}\]

From (15.32) we find that for any integer \( n \),
\[
oc(\gamma + 2\pi n; t) = \sum_{m=0}^{\infty} p \left[ \cos 2\pi (\gamma + \pi n) \right] u(\gamma; t)
= \sum_{m=0}^{\infty} \delta(\gamma + 2\pi m; t) = o(x; t) \tag{15.35}\]

Thus \( o(x; t) \) is periodic with period \( 2\pi \).

Now we may solve \( o(x, t) \) over the interval of one period \(-\pi \leq \gamma \leq \pi \), with the following conditions:
\[
o(0, 0) = \varphi_0 + 2\pi n, \quad -\pi \leq \gamma \leq \pi
\]
\[
o(\pi, t) = o(-\pi, t), \text{ for all } t \tag{15.36}\]
and
\[
\int_0^\infty o(x, t) dx = 1, \text{ for all } t \tag{15.37}\]

We will now solve (15.33) in the steady state, corresponding to the condition
\[
\int_0^\infty o(x; t) dx = 0, \text{ as } t \to \infty \tag{15.39}\]
Thus we write
\[ u(\psi) = \lim_{r \to \infty} u(\psi, r) \]  \hspace{1cm} (15.40)
and (15.33) is written as
\[ -\frac{\partial}{\partial \psi} ((\Omega - AK \sin \psi) u(\psi)) + \frac{KN^2}{4} \frac{\partial^2 u(\psi)}{\partial \psi^2} = 0. \]  \hspace{1cm} (15.41)

Putting
\[ e = \frac{4A}{KN^2} \]  \hspace{1cm} (15.42)
and
\[ \beta = \frac{4\Omega}{KN^2} \]  \hspace{1cm} (15.43)
we get from (15.41)
\[ 0 = -\frac{\partial}{\partial \psi} ((\beta - e \sin \psi) u(\psi)) + \frac{\partial^2 u(\psi)}{\partial \psi^2} \]  \hspace{1cm} (15.44)
Conditions (15.36) through (15.38) remain the same except they now appear
\[ u(\pi) = u(-\pi), \]  \hspace{1cm} (15.45)
and
\[ \int_{-\pi}^{\pi} u(\psi) \, d\psi = 1 \]  \hspace{1cm} (15.46)
Integrating once we get,
\[ \frac{\partial u(\psi)}{\partial \psi} + (\alpha - \beta \sin \psi) u(\psi) = D \]  \hspace{1cm} (15.47)
where \( D \) is a constant to be determined from the boundary condition (15.45). Integrating (15.47) again we find that
\[ u(\psi) = C \exp \left( (\alpha \cos \psi + \beta \psi) \right) (1 + D) \]
\[ \int_{-\pi}^{\pi} \exp \left( (\alpha \cos x + \beta x) \, dx \right) \], \[-\pi \leq \psi \leq \pi \]
\[ (15.48) \]
using (15.45), one finds
\[ D = \frac{\exp(-2\pi c) - 1}{\int_0^\pi \exp(-\pi \xi \cos x - \xi c) \, dx} \]  

(15.49)

\[ \omega(\phi) = \frac{\exp(\xi \cos \phi + i\phi)}{4\pi \exp(-\xi c) I_0(\xi)} \int_0^{2\pi} \exp(-\xi \cos x + i\phi) \, dx \]  

(15.50)

where \( I_0 \) is the modified Bessel function of the first kind and purely imaginary order. It can be shown [5] that

\[ I_0(\xi) = \frac{\sin \xi + \xi \cos \xi}{\xi^2} \]  

(15.51)

when the detuning (2) is zero, i.e., \( \beta = 0 \), the pdf simplifies to

\[ \omega(\phi) = \frac{\exp(\xi \cos \phi)}{2\pi I_0(\xi)} \]  

(15.52)

Therefore, the variance of the phase error in a first order PLL with

\[ \sigma^2 = \frac{1}{\pi} \exp(\xi \cos \phi) \, d\phi \]  

(15.55)

Using (15.52) and Jacobi-Anger formula,

\[ \exp(\xi \cos \phi) = \sum_{n=0}^{\infty} (-1)^n I_n(\xi) \cos n\phi \]

it is easily seen that

\[ \sigma^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n I_n(\xi)}{n^2 I_0(\xi)} \]  

(15.54)

Now if the input carrier strength is large compared to the noise power, i.e., when \( a \) is large, \( \phi \) is not large and we can use

\[ \cos \phi \approx 1 - \phi^2/2 \]

and

\[ I_a(\xi) \approx \frac{\sin \xi}{\sin \xi} \]
Solving these in (15.52) one gets

$$\omega(\varphi) = \frac{\exp\left(-\varphi^2/2\right)}{\sqrt{2\pi/n}}, \quad n \gg 1$$

(15.55)

i.e.,

$$\beta_3^2 = 1/n$$

(15.56)

This indicates for large values of the input carrier-to-noise ratio, the $\beta_3^2$ becomes asymptotically Gaussian with variance $1/n$. Now in the other extreme case when the input carrier-to-noise goes to zero, i.e., $n = 0$, one finds that

$$\beta_3^2 = \pi/3$$

(15.57)

This is the variance of a random variable that is uniformly distributed from $-\pi$ to $\pi$. That is,

$$\omega(\varphi) = \frac{1}{2\pi}, \quad -\pi \leq \varphi \leq \pi$$

(15.58)

Another parameter, that is of importance, is the cumulative steady state probability distribution, defined as

$$P(\varphi_1 < \varphi < \varphi_2) = \int_{-\pi}^{\varphi_2} \omega(\varphi) d\varphi, \quad 0 \leq \varphi_1 \leq \pi$$

(15.59)

It indicates the probability that the loop phase error lies within $\varphi_1$. The definition of the steady state cumulative probability distribution may be used to find the chances of losing lock of the PLL. In order to compute this, one has to find the probability that the phase difference $\varphi$ exceeds the phase stretch between $-\pi/2$ and $\pi/2$. Therefore, the probability of losing lock is

$$P(\pi/2 < \varphi < \pi) = P(\pi < \varphi < -\pi/2) = P_e$$

i.e.,

$$P_e = \int_{\pi/2}^{\pi} \omega(\varphi) d\varphi$$

(15.60)

Using the expression for $\omega(\varphi)$, one can easily show that

$$P(\varphi_1 < \varphi_0) = \frac{1}{n} + \frac{2}{n} \sum_{k=1}^{n/2} l(k) \cos \frac{\varphi_0}{n}$$

(15.61)
\[ P_{\phi} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{2m+1} \]  

(15.62)

Plots of \( P(\phi < \phi_i) \) and \( P_\phi \) are shown in Figs. 15.6 and 15.7 respectively.

- Figure 15.6: Variation of the probability density function of a first order PLL with the phase error, (a) its loss of lock probability.
- Figure 15.7: Cumulative probability density function of the phase error of a first order PLL, operating without initial alignment.
We now return to (15.56) and write again the phase error variance, replacing $\alpha$ from (15.42). Thus
\[
\overline{\sigma^2} = \frac{KN_t}{4\Delta} = \left(\frac{AK}{4}\right)\frac{N_t}{\Delta^2}
\]  
(15.63)

Recalling that $AK/4$ represents the loop noise bandwidth, $B_n$, we rewrite (15.62) as
\[
\overline{\sigma^2} = B_n N_t
\]  
(15.64)

Before we conclude this part of our discussion we ask: What are the average values of the instantaneous output frequency $\bar{\phi}$ and the mean square value of $\dot{\phi}$? These are calculated in the following way, we use (15.24) for $\Omega = 0$,
\[
\bar{\phi} = \frac{d\phi}{dt} = -AK\sin\phi - KN(t)
\]  
(15.65)

Therefore,
\[
\bar{\phi} = -\int AK\sin\phi \, a(\phi) \, d\phi
\]  
(15.66)

Since $N(t)$ is a Gaussian noise with mean zero. That is, we rewrite (15.66) using (15.52)
\[
\bar{\phi} = -\frac{AK}{2\pi J_d(0)} \int \sin\phi \exp(a \cos\phi) \, d\phi = 0
\]  
(15.67)

and the mean square value of $\dot{\phi}$ is
\[
\overline{\dot{\phi}^2} = AK^2 \sigma^2 + K^2 \delta^2
\]  
(15.68)

where $\delta^2$ is the variance of $N(t)$, and
\[
\sin^2\phi = \frac{\overline{\phi}}{\delta} \sin^2\phi \, a(\phi) \, d\phi
\]
\[
= \frac{1}{4\pi J_d(0)} \int (1 - \cos \Delta \omega) \sum_{n=0}^{\infty} J_n(\Delta) \cos n\omega \, d\phi
\]
\[ \varphi = \frac{1}{2} \frac{I_0(a)}{4L_0(a)} \]  
\[ \text{(15.69)} \]

i.e.,

\[ \varphi = \frac{AK^2}{2} - \frac{AKV_0(I_a)(s)}{2L_0(s)} + \frac{K^2}{2} \varphi \]  
\[ \text{(15.70)} \]

When \( a \) is large, this expression reduces to

\[ \varphi \approx \frac{AK^2}{2} = \frac{AK}{2}(\frac{1}{Q}) \]  
\[ \text{(15.71)} \]

When there is a finite frequency error, \( \beta \neq 0 \), the calculation becomes quite involved [5]. However, the mean value of \( \varphi \) may be calculated using \( u(\varphi) \) as

\[ u(\varphi) = W \exp(a \cos \varphi) \frac{I_0(a)}{s} \]

\[ + 2 \sum_{n=1}^{\infty} \frac{(-1)^n I_n(a)}{n^2 + 2} + W \left( \beta \cos \alpha + n \sin \alpha \varphi \right) \]  
\[ \text{(15.72)} \]

where

\[ W = \frac{\sinh(\beta \lambda)}{2\lambda I_{0}(\lambda)} \]  
\[ \text{(15.73)} \]

The variation of \( \varphi \) against \( \beta \) is shown in Fig. 15.8. The curve of \( \varphi \) versus \( \beta \) for \( a = \infty \), corresponds to the case of PLL without any noise at the input.

Fig. 15.8. Variation of the average beat angular frequency with the detuning for different values of the CNR.
15.5 Cycle Slipping Phenomena

For the sake of simplicity let us consider the characteristics of PLL without initial frequency error between the VCO and the synchronizing signal. We have seen in such a case and in the absence of additive random noise, the loop will be locked to the incoming signal without any steady state phase error. Now if the incoming signal becomes corrupted with additive random noise, there will be phase jitter around $\varphi = 0$. Since the noise is a random process, at times its amplitude will be large and it may increase the phase error beyond $\pm \varphi / 2$ radians or $ \pm 90$ degrees. Once the phase error goes beyond $\pm 90$ degrees, the loop is thrown out of lock. But after this, the loop cannot remain out of lock for ever for two reasons, viz. (i) the noise is a random process, meaning that instantaneous amplitude does not remain large for all the time, and (ii) the presence of the synchronizing signal will always act as a restoring force to bring the loop back to the locked state. Because of these reasons, the loop will be ultimately brought back to lock. Now the locking will not be at $\varphi = 0$, but it will be around either $\varphi = \pi / 2$ or $\varphi = - \pi / 2$. In doing this, the loop gains or drops a cycle relative to the incoming signal.

This sort of phase excursion of the PLL is similar to the random walk problem with an absorbing boundary at some point $x = x_0$. The presence of the absorbing wall at $x = x_0$ will indicate an end of the random walk journey as soon as the man reaches the wall. In the case of the phase motion, the absorbing boundary is located at $\varphi = \pm \pi / 2$. A typical phase trajectory is shown in Fig. 15.9, starting at $\varphi = 0$.

We now desire to find the average time which will elapse before either of the boundary is reached. We proceed in the following way [1]. As long as the loop phase error lies within the limits of $\varphi = \pm \pi / 2$, the density function of $\varphi$, i.e., $u(\varphi, t)$, which is now denoted by $q(\varphi, t)$, satisfies the F – P equation, (15.30) [11 = 0]

$$\frac{\partial q(\varphi, t)}{\partial t} = \frac{\partial}{\partial \varphi}[(A \varphi \sin \varphi) q(\varphi, t)] + \frac{2b \varphi}{\pi^2} \frac{\partial^2 q(\varphi, t)}{\partial \varphi^2}$$

with the initial condition

$$q(\varphi, 0) = \delta(\varphi)$$
The reason for distinguishing $q(\phi, t)$ from $u(\phi, t)$ is this. As soon as $\phi$ reaches $2\pi$, for the first time, the process is instantly terminated. That is,

$$q(2\pi, t) \sim q(-2\pi, t) = 0, \text{ for all } t$$  

(15.75)

The probability that $\phi$ has not yet reached $2\pi$ at time $t$ is denoted by

$$\Psi(t) = \int_{2\pi}^{\phi(t)} q(\phi, t) \, d\phi$$  

(15.76)

Obviously,

$$q(\phi, t) = 0 \text{ for all } |\phi| \geq 2\pi \text{ at all } t$$  

(15.77)

since the phase process is terminated as soon as $|\phi|$ becomes $2\pi$. Because of (15.75), the limit of integration of (15.76) could be increased to infinite, i.e.,

$$\Psi(t) = \int_{0}^{\phi(t)} q(\phi, 0) \, d\phi < 1$$  

(15.78)

Note that $\Psi(t)$ is the probability that $|\phi|$ remains within $2\pi$. That is, the loop has not gone out of lock in the time interval from 0 to $t$. 

Fig. 15.9. A typical phase trajectory illustrating loss of lock of a PLL in a noisy environment.
Thus, it is a monotonically decreasing function of time. Therefore, the probability that a cycle has slipped is \(1 - \Psi(t)\). Hence, the probability density function of time required for \(\varphi\) to reach \(2\pi\) for the first time is

\[
\lim_{\Delta t \to 0} \frac{\Psi(t) - \Psi(t + \Delta t)}{\Delta t} = -\frac{\partial \Psi}{\partial t} \tag{15.79}
\]

Therefore, the expected time to slip a cycle is

\[
T = \int_0^\infty \frac{\Psi(t)}{t} \, dt = -\left[ t\Psi(t) \right]_0^\infty + \int_0^\infty \frac{\Psi(t)}{t} \, dt \tag{15.80}
\]

Again \(\Psi(0) = 1\), and \(\Psi(t) \approx t^{-1+\epsilon}\). Otherwise the integral would become infinite. Thus

\[
r\Psi(t) \bigg|_0^\infty = 0 \tag{15.81}
\]

Therefore,

\[
T = \int_0^\infty \Psi(t) \, dt = \int_0^\infty \frac{\Psi(t)}{t} \, dt + \int_0^\infty q(v, t) \, dt \tag{15.82}
\]

Let us now define the following quantity

\[
\mathcal{Q}(\varphi) = \int_0^\infty q(v, t) \, dt \tag{15.83}
\]

where

\[\mathcal{Q}(2\pi) = \mathcal{Q}(-2\pi) = 0\] \hspace{1cm} (15.84)

Again we note that

\[
q(v, \infty) = 0 \quad \text{and} \quad q(v, 0) = \xi(v) \] \hspace{1cm} (15.85)

Now we integrate both sides of (15.73a) with respect to \(t\) over the infinite interval,

\[
\frac{d}{dt} \left( \frac{\mathcal{Q}(v) \sin \varphi}{\varphi} \right) = \frac{KN}{4} \frac{d^2 \mathcal{Q}(v)}{d\varphi^2} \tag{15.86}
\]
Putting the boundary condition \((15.85)\) and integrating \((15.86)\) once with respect to \(x\), we get

\[
C - \psi(\varphi) = (AK \sin \varphi) \psi(\varphi) + \frac{K N_A}{4} \frac{d\psi(\varphi)}{d\varphi} \quad (15.87)
\]

where \(C\) is a constant and \(U(\varphi)\) is a unit step function. On integration \((15.87)\) becomes

\[
\psi(\varphi) = D \exp(\alpha \cos \varphi) + \frac{\exp(\alpha \cos \varphi)}{\gamma} \int_{-\pi}^{\varphi} \exp(-\alpha \cos \chi) [C - U(\chi)] d\chi
\]

\[
(15.88)
\]

where

\[
\gamma = \frac{2 K N_A}{4}
\]

Applying the boundary condition \((15.84)\) one finds

\[
D = 0 \text{ and } C = \frac{1}{2}
\]

Thus

\[
\psi(\varphi) = \frac{\exp(\alpha \cos \varphi)}{\gamma} \int_{-\pi}^{\varphi} [ \frac{1}{2} - U(\chi) ] \exp(-\alpha \cos \chi) d\chi
\]

\[
(15.90)
\]

Therefore, from \((15.82)\) and \((15.83)\) one gets

\[
T = \int_{\varphi}^{\pi} \psi(\varphi) d\varphi
\]

\[
= \frac{1}{2} \int_{-\pi}^{\varphi} d\varphi \int_{-\pi}^{\varphi} \exp(-\alpha \cos \varphi) [ \frac{1}{2} - U(\varphi) ] d\chi
\]

\[
(15.91)
\]

Referring to Fig. 15.10 and using \((15.91)\)

\[
T = \int_{\varphi}^{\pi} d\varphi \int_{-\pi}^{\varphi} \exp(-\alpha \cos \varphi - \cos \varphi) d\chi
\]

\[
(15.92)
\]

Using again the Jacobi–Anger relation, viz.,
\[ \exp(-\alpha \cos \theta) = \sum_{n=-\infty}^{\infty} (-1)^n I_n(\alpha) \cos n\theta \]

\[ T = \frac{1}{\gamma} \int_{0}^{2\pi} \left[ I_0(\theta) + 2 \sum_{m=1}^{\infty} I_m(\theta) \cos m\theta \right] \cos \xi d\xi \]

\[ X[I_0(\xi) + 2 \sum_{n=1}^{\infty} (-1)^n \cos n\xi] d\xi \]

\[ = \frac{2\pi}{\gamma} I_0(\alpha) = \frac{\pi \alpha}{2\gamma} I_0(\alpha) \]

\[ \text{where} \]

\[ R_n = \frac{AK}{\alpha} \]

Hence the frequency of skipping cycles is given by

\[ f = \frac{2\alpha R_n}{\pi I_0(\alpha)} = \frac{AK}{\pi I_0(\alpha)} \]

At high signal-to-noise ratio (\( \alpha \gg 1 \)), we have

\[ F = \frac{AK}{\alpha} \exp(-2\alpha) \]

and

\[ T = \frac{\pi}{AK} \exp(2\alpha) \]

15.6. Nonlinear Analysis of Second Order Loops

Let us consider a P.I. incorporating a proportional plus integrating filter having a transfer function
\[ F(x) = \frac{1 + E_T s}{1 + sT} \]

then the governing phase equation of the loop (\( \Omega = 0 \)) is given by (cf. 9.34) [\( \Omega = 0 \)]

\[ T \frac{d^2 \phi}{dT^2} + \frac{d\phi}{dT} = -K [A \sin \phi + N(t)] \]

\[ -F_T \frac{\phi}{dt} [A \sin \phi + N(t)] \quad (15.97) \]

In order to avoid the term, incorporating the derivative of the white noise \( N(t) \), we make the following substitution.

\[ q(t) = \dot{z} + F_T \frac{d\phi}{dT} \quad (15.98) \]

Then (15.97) reduces to

\[ T \frac{d^2 q}{dT^2} + \frac{d q}{dT} + F_T \frac{d q}{dT} \left[ T \frac{d \phi}{dT} + \frac{d \phi}{dT} \right] \]

\[ = -K [A \sin \phi + N(t)] - F_T \frac{d \phi}{dT} [A \sin \phi + N(t)] \]

\[ (15.99) \]

Since this differential equation is the sum of two terms, one of which is the derivative of the other (of course with a multiplying factor \( F_T \)), the solution of (15.99) will be the same as that of the following differential equation:

\[ T \frac{d^2 q}{dT^2} + \frac{d q}{dT} = -K [A \sin \phi + N(t)] \]

\[ (15.100) \]

Putting

\[ z = \phi \]

and

\[ \frac{d z}{dt} = \dot{z} \]

we obtain the following equations out of (15.100).

\[ \frac{d^2 \phi}{dT^2} + \dot{z} + AK \sin (\dot{z} + E_T \dot{z}) \phi = -K \phi(t) \]

\[ (15.101) \]

\[ (15.102) \]

\[ (15.103) \]
From the equations it is evident that \( y_d(t), y_q(t) \) is a Vector Markov process. The joint probability density function of the two variables \( y_d(t) \) and \( y_q(t) \) satisfies the following second order F - F equation

\[
\begin{align*}
\frac{\partial^2 E(y_d, y_q)}{\partial t^2} & = - \frac{1}{\frac{\partial^2}{\partial t^2}} \left[ A_0(y_d, y_q) \rho(y_d, y_q, t) \right] \\
& \quad + \frac{1}{2} \frac{\partial^2}{\partial y_d \partial y_q} \left[ A_0(y_d, y_q) \rho(y_d, y_q, t) \right] \\
& \quad + \frac{1}{2} \frac{\partial^2}{\partial y_d \partial y_q} \left[ A_0(y_d, y_q) \rho(y_d, y_q, t) \right] \\
& \quad + \frac{1}{2} \frac{\partial^2}{\partial y_d \partial y_q} \left[ A_0(y_d, y_q) \rho(y_d, y_q, t) \right]
\end{align*}
\]

(15.104)

\[
A_d(y_d, y_q) = \log \frac{E(y_d, y_q)}{\Delta t}
\]

(15.105)

\[
A_d(y_d, y_q) = \lim_{\Delta t \to 0} \frac{E(y_d, y_q)}{\Delta t}
\]

(15.106)

Therefore, from (15.120) and (15.103) we get

\[
\begin{align*}
A_0(y_d, y_q) & = y_t \\
A_d(y_d, y_q) & = y_d + 4K \sin (y_d + TF_s y_q) \\
A_d(y_d, y_q) & = A_d(y_d, y_q) = A_d(y_d, y_q) = 9 \\
A_d(y_d, y_q) & = \frac{K N y_t^2}{2}
\end{align*}
\]

(15.107)

(15.108)

(15.109)

Thus, inserting these coefficients in (15.104) one finds

\[
\frac{\partial \rho}{\partial t} = - \frac{\partial \rho}{\partial y_d} \left[ 4K \sin (y_d + TF_s y_q) \right] + \frac{K N y_t^2}{2} \rho
\]

(15.111)

where \( \rho \) denotes \( \rho(y_d, y_q, t) \) and the initial condition

\[
\rho(y_d, y_q, t) = \rho_0(y_d, y_q, t)
\]

In the steady state, (15.111) reduces to

\[
\frac{\partial \rho}{\partial t} = - \frac{\partial \rho}{\partial y_d} \left[ 4K \sin (y_d + TF_s y_q) \right] + \frac{K N y_t^2}{2} \rho
\]

(15.112)

In most cases, this partial differential equation has no analytical solution because of its nonlinear character. Eqs. (15.112) can be
solved only when \(P_0 = 0\), i.e., when the filter transfer function is given by

\[
F(t) = \frac{1}{1 + xT}
\]

In such a situation, referring to (15.98) and (15.101) one finds that \(\gamma_1 = \dot{\phi}\), and \(\gamma_2 = \ddot{\phi}\). Thus we rewrite (15.112) as

\[
\dot{\theta} \frac{\partial \tilde{\theta}(\phi, \tilde{\phi})}{\partial \phi} = \frac{1}{T} \frac{\partial}{\partial \phi} \left[ \frac{\partial}{\partial \phi} \left( 2K \tilde{\theta} + 2K \tilde{\phi} \right) \right]
\]

Since the coefficient in the above equation is periodic in \(\phi\), we write the general solution as (see (15.4))

\[
u(\phi, \tilde{\phi}) = \frac{2}{T} \pi \left( \phi + 2\pi n, \tilde{\phi} \right)
\]

Thus (15.113) reduces to

\[
\dot{\theta} \frac{\partial \tilde{\theta}(\phi, \tilde{\phi})}{\partial \phi} = \frac{1}{T} \frac{\partial}{\partial \phi} \left[ \frac{\partial}{\partial \phi} \left( 2K \tilde{\theta} + 2K \tilde{\phi} \right) \right]
\]

It is easily verified that the solution is given by

\[
u(\phi, \tilde{\phi}) = C_1 \exp \left( \frac{4\phi}{K^2} \cos \phi \right) \times C_2 \exp \left( -\frac{\tilde{\phi}^2}{2K^2} + \frac{\phi^2}{2K^2} \right)
\]

Thus we find that \(\phi\) and \(\tilde{\phi}\) are statistically independent and we write

\[
u(\phi) = \nu(\phi) \cdot \nu(\phi)
\]

where

\[
u(\phi) = C_1 \exp \left( \alpha \cos \phi \right)
\]

\[
u(\phi) = C_2 \exp \left( -\phi^2 / 2 \alpha^2 \right)
\]

\[
\alpha = \frac{4K^2}{K^2}
\]

and

\[
\alpha = \frac{K^2}{4T}
\]
412 Phase Lock Theories and Applications

The constants $C_2$ and $C_3$ are evaluated from the following normalizing conditions

$$\int_{-\infty}^{\infty} \omega(q) dq = 1$$  \hspace{1cm} (15.121)

and

$$\int_{-\infty}^{\infty} \omega(q) dq = 1$$  \hspace{1cm} (15.122)

From these conditions one gets

$$C_2 = \frac{1}{2\pi L (\omega)}$$ 

and

$$C_3 = \frac{1}{2\pi T}$$

Hence

$$\omega(q) = \frac{1}{2\pi L (\omega)} \exp(a \cos \phi)$$  \hspace{1cm} (15.123)

and

$$\omega(q) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left( -\frac{q^2}{2\sigma^2} \right)$$  \hspace{1cm} (15.124)

Comparing these with the pdf of $q$ for the first order loop, one finds that $\omega(q)$ is the same for both the loops. But, the variance of the frequency error in the second order loop is intensely related to the time constant of the filter network. In passing we again emphasize that accurate expressions for $\omega(q)$ and $\omega(q)$ cannot be given for PLL's incorporating filter networks. $(1 + F \gamma T)/(1 + S T)$ or $(1 + S T) e^{s T}$, However, approximate expressions have been given in certain works (1, 5). For this reason, sometimes the technique of quasi-linearization is used. This we discuss in the following section.

15.7 Quasi-Linearization Technique

This technique replaces a nonlinear function $[f(x)]$ of the Gaussian random variables $x$ by a linear function of $x$ such that the mean square error between the actual output and the approximate output is minimum. Representing the approximate output by $m + K x$, $m$
being the mean or the average output and $K$, the equivalent gain, one finds that the mean square error is given by
\[
\langle \varepsilon^2 \rangle = \langle [f(x) - m - Kx]^2 \rangle = \langle f(x)^2 \rangle + m^4 + K^2 \langle \varepsilon^2 \rangle - 2m \langle f(x) \rangle \\
+ 2mK \langle x \rangle - 3K \langle x f(x) \rangle
\]
(15.125)

Assuming that $x(t)$ is a Gaussian random variable with zero mean and probability density function $p(x)$ it is easy to show that
\[
\langle x \rangle = 0
\]
\[
\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x) \, dx
\]
\[
\langle x f(x) \rangle = \int_{-\infty}^{\infty} x f(x) p(x) \, dx
\]

Minimizing $\langle \varepsilon^2 \rangle$ with respect to both $K$ and $m$ one finds that
\[
m = \int_{-\infty}^{\infty} f(x) p(x) \, dx
\]
(15.126)
and the equivalent gain
\[
K = \frac{\int_{-\infty}^{\infty} x f(x) p(x) \, dx}{\int_{-\infty}^{\infty} x^2 p(x) \, dx}
\]
(15.127)

Thus we write
\[
f(t) \approx m + Kx
\]
(15.129)
where $m$ and $K$ are given by (15.126) and (15.127) respectively.

5.8 Acquisition Analysis in a Noisy Environment

In this section we present the locking analysis of a PLL in a noisy environment based upon the previous analysis (6.8). Phase governing
equation of a PLL in the presence of an additive white Gaussian
noise is given by
\[ \frac{d\gamma}{dt} = \Omega - AK(\delta)F(0)j_{r}(m) \sin \gamma \]  
(15.129)

where the symbols have their usual significance as stated elsewhere
in the text. Following the procedure of section 10.6, we assume the
solution of (15.129), in the heating condition, as
\[ \gamma = \omega + u(t) + m(t) \sin \omega t + \eta \]  
(15.130)

where \( \eta \) denotes the noise modulation of the VCO due to the pre-

cence of the incoming noise \( \omega(t) \) and \( m(t) \) are slowly varying func-
tions of time. These are responsible for gradual change of the average
\( \eta \)-detector output voltage, leading to ultimate synchronization of the
loop. Substituting the assumed solution is (15.129)
\[ \omega + \frac{d\gamma}{dt} + \pi \cos \omega t + \sin \omega t \frac{dm}{dt} + \varphi \]
\[ = \Omega - KF(p) [A \cos \varphi \sum_{m} j_{m}(m)] \]
\[ \sin [(1 + \theta) \omega + \gamma] \]
\[ + A \sin \varphi \sum_{m} j_{m}(m) \cos [(1 + \theta) \omega + \gamma] + N(t) \]  
(15.131)

Using a linearization technique and assuming that \( \varphi \) is a Gaussian
noise with variance \( \Phi \), we approximate \( \sin \varphi \) and \( \cos \varphi \) by the

Therefore, by applying the method of harmonic balance and using
(15.131), (15.132) and (15.133) we get
\[ \frac{d\gamma}{dt} = \Omega - \omega + AK(\delta) F(0) j_{r}(m) \sin \gamma \]  
(15.134)

\[ \frac{dm}{dt} = -AK(\delta) F(0) j_{r}(m) \cos (\gamma - \gamma) - j_{m}(m) \cos (\gamma + \gamma) \]  
(15.135)

\[ m = -AK(\delta) F(0) j_{r}(m) \cos (\gamma - \gamma) - j_{m}(m) \sin (\gamma + \gamma) \]  
(15.136)
\[ y = LF(x) \] (15.137)

\[ \Omega = AK(b) \langle F(0) \rangle \] (15.139)

The frequency pulling time is obtained as

\[ T_F = \int_{y_0}^{y} \frac{dy}{\Omega - AK(b) \sin \gamma} \] (15.141)

where

\[ \Omega = \Omega - 0.75 AK(b) \langle F(0) \rangle \]
\[ AK(b) = 0.25 AK(b) F(0) \]
\[ \gamma = -\varphi + \eta(\Omega) \]

\[ \gamma = \arcsin (\Omega AK(b) F(0)) \]

and the value of \( \beta \) is calculated from (15.137) [in 1] and is given by

\[ \beta = \frac{1}{2\pi} \int \frac{KF(x)}{\left( \frac{1}{4} + \frac{1}{2} AK(b) \cos (\eta(\Omega)/2) F(0) \right)^2} S_\eta(\Omega) \, d\Omega \] (15.142)

where \( S_\eta(\Omega) \) is the spectral density of \( \eta(\Omega) \). In (15.142) the value of \( \gamma \) has been replaced by its average \( \overline{\varphi} + \gamma/2 \), meaning that the final value of \( \gamma \) is nearly \( \pi/2 \).

Example:

Let

\[ F(x) = \frac{1}{1 + x^2} \]

\[ \cos \left( \frac{\eta(\Omega)/2}{2} \right) = \frac{1 - \cos \frac{\eta(\Omega)}{2}}{2} \]
Putting

$$\Delta_{p}\sin \omega t = \frac{1}{\sqrt{2}} \left(1 + \sqrt{1 + (2\Delta F)^2} \right)^{\frac{1}{2}}$$

(15.143).

Putting

$$K_0 = \frac{1}{2} K(0) \cos \left[\frac{\pi}{2} (\Delta / 2)\right]$$

and taking $$S_{\delta}(u) = N_0 / 2$$, one rewrites (15.142) as

$$\psi = \frac{1}{2\pi} \left(\frac{K_0}{\Delta F}\right) \int \frac{AK_{\delta}}{s^2 + \Delta F_{\delta}} \sin \frac{N_0}{\Delta F} ds$$

(15.144).

i.e.,

$$\psi = \left(\frac{K_0}{\Delta F}\right) \frac{N_0}{\Delta F} \frac{AK_{\delta}}{4}$$

(15.145).

The solution of (15.139) along with (15.143) and (15.145) requires a graphical technique.

15.9 Response of a PLL to a Test-tone Modulated Signal Corrupted with AWGN

Let us suppose that the incoming signal is in tune with the center frequency of the VCO. Denoting the incoming FM signal and the output of the VCO as $$\sqrt{2}A \sin (u_0 t + \Delta_0 \sin \omega t)$$ and $$\sqrt{2}A \cos \left[u_0 t + \Delta_0 \sin \omega t - \gamma_0\right]$$ respectively, the output of the phase detector in the presence of AWGN is given by

$$v_p = K_0 K_{\delta} (A \sin (u_0 t + \Delta_0 \sin \omega t) + N(t))$$

(15.146).

$$\gamma_0$$ denotes the noise phase modulation of the VCO. The corresponding equivalent analytical block diagram of the PLL is shown in Fig 5.11. Thus referring to Fig 5.11, one finds that the input to

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**Fig. 5.11.** The equivalent analytical representation of a phase locked demodulator.
the sinusoidal nonlinearity consists of a test-tone signal and a random component \( \eta \). Thus the response of the phase detector to the signal component will be different from the response to the noise component. And these will depend on the signal-to-noise ratio at the input to the phase detector. Exact nonlinear analysis of a higher order PLL in such a situation is not possible and we adopt the technique of quasi-linearization techniques. Thus assuming the character of \( \eta \) to be Gaussian in nature with variance \( \sigma^2 \), we treat the phase detector as a combination of two equivalent linear channels—one for the signal and the other for the noise—having equivalent gains \( g_s \) and \( g_n \), respectively. Thus we break the original analytic loop of Fig. 15.11 into two equivalent linearized loops—one for the signal and the other for the noise as shown in Fig. 15.12. Denoting \( (\delta_1 - \delta_0) \) by \( \phi \) and referring to Chapter 13, we write the expressions for the equivalent gains as

\[
\frac{2}{\pi} \int \mathcal{F}(\phi_0) \cdot J_{1}(m\phi) \exp \left( -\frac{b^2 \phi^2}{2} \right) d\phi
\]

and

\[
\frac{1}{2\pi} \int \mathcal{F}(\phi_0) \cdot J_{0}(m\phi) \exp \left( -\frac{b^2 \phi^2}{2} \right) d\phi
\]

where

\[
\mathcal{F}(\phi) = F_s(\phi_0) + F_n(\phi_0)
\]
\[ F_\varepsilon(\varphi) = \sin \varphi, \quad \varphi > 0 \] (15.149)  
\[ F_\varepsilon(\varphi) = \sin \varphi, \quad \varphi \leq \varepsilon \] (15.150)  
and  
\[ F_\varepsilon(\varphi) = \sin \varphi, \quad \varphi > \varepsilon \] (15.151)  

For the sake of convenience we approximate the sinusoidal non-linearity by the following empirical relation
\[ \sin \varphi = -0.993 \varphi - 0.655 \varphi^2 + 0.092 \varphi^3, \quad -\pi \leq \varphi \leq \pi. \] (15.152)

Figure 15.13 shows the plots of \( \sin \varphi \) and the equivalent empirical relation with \( \varepsilon \). In the present case, since the frequency error is zero, the expressions for the equivalent gains reduce to
\[ g_i = \frac{4}{\lambda t} \int_{\lambda t} \sin(\varphi) \sin(\varphi) \exp \left( -\frac{\varphi^2}{2} \right) d\varphi. \] (15.153)

and  
\[ g_s = \frac{2}{\lambda t} \int \sin(\varphi) \cos(\varphi) \exp \left( -\frac{\varphi^2}{2} \right) d\varphi. \] (15.154)

where \( F_\varepsilon(\varphi) \) comes out to be (cf. 15.151)
\[ F_\varepsilon(\varphi) = \sum_{n=1,3,5,\ldots} \left( \frac{\varphi + 1}{\varphi} \right)^n \] (15.155)
\[ a_n = 0.093, a_1 = -0.153 \text{ and } a_2 = 0.025. \]

From (15.152) through (15.154) one finds that
\[ g_i = 0.993 - 0.459 \varphi^2 (1 + 0.243 \varphi^2) + 0.078 \varphi^4 (1 + 0.58 \varphi^2) \] (15.156)

and  
\[ g_s = 0.993 - 0.459 \varphi^2 (1 + 0.58 \varphi^2) + 0.078 \varphi^4 (1 + 0.58 \varphi^2). \] (15.157)

Now refer to Fig. 15.12 and assume
\[ F_{\delta}(\delta) = \frac{1 + F_\varepsilon \delta}{1 + \delta}, \] and find that the variances for the modulation and noise phase errors are given by
\[ \text{Var} = \frac{M^2}{2}, \quad \delta_1 = \frac{1}{2} (1 - F_\varepsilon(\delta))^2. \] (15.158)
and

\[ y(t) = \frac{N_{0}}{(4\pi)^{1/2}} \int_{-\infty}^{\infty} |H_{a}(f)| \frac{1}{f} \, df \]

where,

\[ H_{c}(f) = \frac{AK_{G_{c}}F_{c}(f)}{s + AK_{G_{c}}F_{c}(f)} \]  
(15.158)

\[ H_{a}(f) = \frac{AK_{G_{a}}F_{a}(f)}{s + AK_{G_{a}}F_{a}(f)} \]  
(15.159)

---

Fig. 15.13 Empirical representation of a sinusoidal function over the interval \(-\pi \text{ to } \pi\) radians and its comparison with the actual sine function.
Putting
\[\alpha^2 = \frac{AK}{T}\]
\[2\delta u = AKF_T + \frac{1}{T}\]
and assuming \(AKF_T T \gg 1, \delta u_T \gg a^3\) and \(x = 1/\sqrt{2}\), we find that the total mean square phase error is given by
\[\delta^2 = \delta_u^2 + \delta^2\]
\[= \frac{A^2 N}{2u_0} \frac{\delta u}{\delta T} + \frac{3N_x u_0}{4\delta T} x\]
where
\[x = 1 + 2a^2\]
(15.160)
(15.161)
Minimizing the total mean square error by setting
\[\frac{d}{d\delta u} (\delta^2) = 0\]
we find the following condition
\[\delta u = \frac{\sqrt{2} A N / \delta T}{3N_x u_0 x}\]
(15.162)
Using this relation, we find that
\[\alpha^2 = \frac{4}{3} \delta_u^2\]
and
\[M^2 = \frac{2}{3} \delta u^2\]
(15.163)
Using (15.160) and (15.162) it is found that
\[\frac{A^2}{N_x f} = 5.237 \sqrt{\frac{A}{f}} (\delta_T^{0.45}) x\]
(15.164)
where \(f\) is the modulating frequency \((2\pi f = \omega_0)\). The above relation indicates the carrier-to-noise ratio in the modulation band when non-linear nature of the phase detector is taken into account.
Now if one assumes that the phase detector characteristic could be replaced by \(\varphi\) instead of \(\sin \varphi\), the expression for the carrier-to-noise ratio would become (follow the same procedure as outlined above).
where $N_f$ is the total phase error variance when linear characteristics of the phase detector is assumed.

We can now find the threshold CNR for the phase locked demodulator by defining the threshold CNR in the following way. It is the value of the CNR at which the actual mean square $\delta^2$ differs from the linearized mean square error by 1.0 dB. That is,

$$\log(\delta^2/\delta_0^2) = 1.0$$

or

$$\delta^2 = 1.259 \delta_0^2$$

(15.165)

To find the value of the threshold CNR, utilizing (15.164), (15.165) and (15.166), we plot $1.259 \delta_0^2$ and $\delta^2$ versus $N_f/\Delta_f \sqrt{\Delta_f}$ as shown in Fig. 15.14. Obviously, the point of intersection of the two curves corresponds to the threshold CNR and this is given by

$$N_f/\Delta_f \sqrt{\Delta_f} = 0.0352$$

(15.167)

In the above we have considered the case of line-to-line modulation. Other cases of modulation can be treated identically except that the expressions for the noise and signal gains are to be evaluated.

For example, in the case of Gaussian noise modulation, which represents FDM signals, referring to Chapter 13 one can readily find the expressions for the signal and noise gain of the phase detector. If we assume that signal and noise components at the input to the phase detector, i.e., at the input to the squaring modulator, are Gaussian in character and have variances $\delta^2$ and $\delta_0^2$ respectively, then it is easy to show that

$$g = g_0 - g_1 = \int_{-\infty}^{\infty} F(u) \rho (u) du$$

i.e.,

$$g = \exp (-\sigma^2/2)$$

(15.168)
where,
\[ \delta^2 = \delta_p^2 + \delta_e^2 \]  
(15.169)

Fig. 15.14. Variation of the mean square phase error of a PLL with note-to-carrier ratio \((N_0 \Delta f \sqrt{\Delta \nu_1 \Delta \nu_2})\). \(\delta_p^2\) denotes the linearized variance and \(\delta_e^2\) denotes the variance calculated on the basis of nonlinear analysis.

Assuming that the spectral density of the FDM signal is of the form
\[ W_r(f) = \frac{S_r}{f} \text{Watt/Hertz} \quad 0 < f < f_c \]  
(15.169a)

it is not difficult to show that the modulation phase error variance is given by
\[ \delta_e^2 = \int_0^f W_r(f) \left| 1 - R_H(f, \nu) \right| df \]  
(15.170)
and the noise phase error variance is given by

$$ H(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(f) \, df $$

(15.171)

where

$$ H(f) = \frac{A_k^2 \Phi(f)}{\pi + A_k^2 \Phi(f)} $$

(15.172)

Hence

$$ H(f) = \frac{2\pi^2 \Phi(f)}{4\pi^2 \Phi(f) + 8\pi^2 \Phi(f)} $$

(15.173)

and

$$ H(f) = \frac{2\pi^2 \Phi(f)}{4\pi^2 \Phi(f) + 8\pi^2 \Phi(f)} $$

(15.174)

Putting

$$ q = \frac{S_{\phi \theta}}{6\pi^2 a^2 \Phi(f)} $$

(15.175)

Minimizing the total mean square phase error with respect to $a_\theta$, one finds the following condition

$$ a_\theta = \frac{1}{\sqrt{2\pi^2 \Phi(f)}} $$

(15.176)

Putting this relation in (15.175), one gets the minimum mean square error as

$$ \hat{a}_\theta^2 = \frac{9913 \times \left( \frac{\Phi(f)}{4\pi^2} \right)^{0.14}}{\Phi(f)} $$

(15.177)

Now defining the mean square frequency deviation as

$$ \Delta f = \frac{\int_{-\infty}^{\infty} \Phi(f) \, df}{\Phi(f)} $$

(15.178)

We find

$$ \Delta f = \frac{S_{\phi \theta}}{2\pi} $$

(15.179)
we define the r.c.s. index of modulation as

\[ m_r = \frac{\Delta}{m} \]  

(15.180)

Inserting \( m \), and the value of \( \phi \) in (15.177), we find

\[ (b_{dr})^{1/2} = 0.7532 \frac{1 + 2g}{2\pi} \frac{\sqrt{m} \omega_{cm}}{\sqrt{g}} \frac{N_0}{\Delta^2} \]  

(15.181)

where

\[ g = \exp \left( -\frac{\Delta^2}{2} \right) \]  

(15.182)

Sometimes, the threshold criterion is taken to be that value of CNR for which the mean square phase error corresponds to 0.25 (radian)\(^2\). Thus putting

\[ \frac{b_{dr}}{g_{dr}} = \frac{1}{4} \]

in (15.182) one finds that the threshold value of the CNR comes out to be

\[ \frac{A^2}{N_{0,dr}} = 33.72 \sqrt{m} \]  

(15.183)

### 15.10 Phase Locked Discriminator Performance

In the preceding section, we have not taken into consideration the effect of cycle slipping of a PLL. The frequency of slipping cycles increases with the decrease of the carrier-to-noise power ratio. We remember that cycle slipping occurs whenever a PLL looses lock. We have seen that whenever a PLL goes out of lock momentarily, (say, about \( \theta = 0 \)) it again comes back to synchronization about \( \theta = \pm 2\pi \). Thus as soon as a PLL slips cycle to cycle, there is a sudden change or some step change in the output phase of the VCO. Again we know that the discriminator output is proportional to the time derivative of the phase. Therefore, the slipping of cycles will produce an impulsive noise component at the output of the discriminator, having an area of \( 2m \). In view of this, it is reasonable to think that the PLL operation under this condition may be represented by the Fig. 15.15. Therefore, the net output of the phase locked discriminator consists of the usual PLL output plus a noise component, given by
Fig. 15.15. Equivalent representation of a phase locked demodulator including the effect of cycle slipping.

\[ N_c(t) = 2 \pi \sum_{k} n_i(t - t_k) \]  \hspace{1cm} (15.184)

where \( n_i \) and \( t_k \) are random variables that depend on the loop parameters. \( n_i \) is Poisson distributed with parameter \( \lambda \). Before we stop in for further analysis we assume that when a PLL loop lock and regain lock, it slips just one cycle. That is, we assume that \( n_i = \pm 1 \) \( \forall \) \( t \).

The output noise power due to this series of noise impulses is given by [106]

\[ N_{in} = 4 \delta \lambda C \]  \hspace{1cm} (15.185)

where \( \delta \lambda \) is the bandwidth of the output filter corresponding to the highest frequency of the modulating signal. Thus to calculate the noise power output due to the spike noise, a knowledge of \( \lambda \), the cycle slipping rate is essential. Unfortunately, \( \lambda \) is not accurately known for a first order loop, where the signal modulation (in time carrier) is absent (cf. 15.96), i.e.,

\[ \lambda = \frac{2B_i}{\sqrt{2} f_{0}(s)} \quad \text{and} \quad \delta \lambda = \frac{4k}{4} \]

which for large value of \( \lambda \) reduces to

\[ \lambda = \frac{4B_i}{\pi} \exp(-2k) \]  \hspace{1cm} (15.186)

\[ \delta \lambda = 4k/N_i \]

However, experimental results indicate that an empirical relation of the following form (11, 12) may be given for a second order type one loop with the damping factor \( 1/\sqrt{2} \),

\[ T = C \delta \lambda \exp(-2k) \]  \hspace{1cm} (15.187)
where the following values of $C$ and $d$ are suggested:

- $C = 1.57$, $d = 1.64$
- $C = 1.0$, $d = 1.6$
- $C = 1.01$, $d = 1.89$

Now we simplify the model of Fig. 15.5. We have already noted that the significant effect of the nonlinear periodic character of the phase detector comes in because of the cycle slipping phenomenon. And the cycle slipping rate is very low, the noise power output, calculated on the basis of the linear model, does vary very little from that calculated on the basis of the nonlinear model of the PLL. In view of this, we linearize the model of Fig. 15.15 and this is shown [9] in Fig. 15.16 for a single tone modulated signal.

\[ \sin(w_0 t + \Delta \sin(\omega_0 t)) \]

*Fig. 15.16. Equivalent analytical representation of the phase-locked demodulator with spike noise.*

Assuming that the loop bandwidth $B_L$ is large compared to $f_m$, the noise power output due to $\Delta \cos^2 f$ is given by

\[ N_{\text{eq}} = \int_0^\infty N_{\Delta \cos^2 f} \, df \]

\[ = N_{\Delta \cos^2 f} \frac{b^3}{6B_L^4} \quad (15.188) \]

Assuming again that $B_L$ is large compared to $f_m$, we find the signal power output is given by

\[ S_{\text{out}} = \frac{(\Delta \omega_0)^2 f}{N_{\text{eq}} + N_{\text{in}}} \quad (15.189) \]

where $(\Delta \omega_0)$ is the maximum frequency deviation.

Therefore, signal-to-noise power output is given by

\[ \text{(SNR)} = \frac{\frac{(\Delta \omega_0)^2 f}{N_{\text{eq}} + N_{\text{in}}}}{N_{\text{eq}} + N_{\text{in}}} \]
That is,

\[
\text{(SNR)} = \frac{3\pi^4 \left( \frac{P}{SW} \right)^2 \text{CNR}}{1 + 24 \left( \frac{P}{SW} \right)^2 \frac{\pi^2}{6} \text{CNR}}
\]  

(15.190)

where

\[
\text{CNR} = \frac{A^2}{2N_0}
\]  

(15.191)

It is interesting to compare this result with that of a standard limiter discriminator (cf. section 3.6). Referring to section 9.2, and putting

\[ N_d(t) = n_1 \cos \Psi_1 + n_2 \sin \Psi_1 \]

and

\[ N_d(t) = n_1 \sin \Psi_1 - n_1 \cos \Psi_1 \]

we find that the output of the bandpass square cut-off filter \( (f_1 - f_2)^2 f_1 + 2f_2) \), can be written as

\[ R_d(t) = \sqrt{2A} \sin (\omega_s t + \Psi_1 + \theta_d) \]  

(15.192)

where

\[ \theta_d = \text{atan} \frac{N_d}{A + N_d(t)} \]  

(15.193)

Refer to the Franel diagram of Fig. 15.17. Note that the point \( P \) denotes the extremity of the vector resulting from the addition of two

[Diagram: The tube triangle at the output of a bandpass limiter (Frenel diagram).]
random vectors \( A + N_q(t) \) and \( N_o(t) \). As such, the pdf followed by \( P \) will be a random one as indicated in Eq. 15.16. The probability that the pdf may encircle the origin \( 0 \) depends on the probability that \( N_q(t) \) or \( N_o(t) \) may exceed \( A \). Therefore, when the strength of the carrier is large compared to that of the noise, the probability of encircling the origin is very small. However, when the carrier-to-noise ratio \( A^2/N_o \) becomes small, this probability increases. As long as the tip of the vector \( OP \) encircles the origin, the resulting phase angle of the net input signal to the discriminator gains an angle of \( 2\pi \) or loose an angle of \( 2\pi \) within a very short interval of time. The discriminator output, which is proportional to the rate of change of this angle, gives positive or negative pulses, known as clicks or spikes. The average number of such clicks (positive or negative) appearing per second is given by (13)

\[
N_+ = N_- = \frac{R}{4\sqrt{2}} \sigma_f(\sqrt{3})
\]

(15.194)

where

\[
\rho = \frac{A^2}{N_oB}
\]

(15.195)

and

\[
\sigma_f(\sqrt{3}) = \int_0^\infty \frac{e^{-\omega^2}}{\sqrt{\omega^2}} d\omega
\]

(15.196)

Further the spectral density of the click noise at the output is given by [13]

\[
S(\omega) = 8m^2N_o + N_o
\]

(15.197)

The total noise power at the output of the limiter-discriminator consist of two components, viz., (1) the ordinary noise of spectral density \( 8m^2N_o/\omega \) and (2) the spike noise of spectral density \( 8m^2N_o \). Therefore, the total noise power at the output is

\[
\langle N^2 \rangle = \frac{K_p^2}{2\pi} \int_0^\infty \frac{a^2}{1 + \frac{\omega^2}{\omega_0^2}} da + \frac{K_p^2}{2\pi} \int_0^\infty \frac{16\pi^2N_o}{1 + \omega^2} d\omega
\]

(15.198)

which for small \( o_o T \) reduces to

\[
\langle N^2 \rangle = \frac{K_p^2N_o}{5\pi^2A^2} + 8\pi mK_p^2N_o
\]

(15.199)

Thus the signal-to-noise power ratio at the output of the LD is
\[
(SNR)_{15} = \frac{3\mu_0 \left( \frac{\beta}{\gamma} \right)}{\gamma}
\]

where \( \gamma = 1 + 4 \sqrt{\frac{\text{erf}(\sqrt{\beta \gamma})}{\gamma \beta}} \)

(15.200)

If the filter preceding the limiter is not an ideal rectangular filter, the average number of clicks in the absence of modulation is given by

\[
N_\gamma \approx \left( \text{erf}(\sqrt{\beta}) \right)
\]

(15.331)

where

\[
F(z) = \frac{1}{2} \int_0^\infty (u - u_0)^2 \left| R(zu) \right|^2 du
\]

(15.207)

\( R(zu) \) is the transfer function of the filter having \( z_0 \) as its centre of symmetry.

Signal-to-noise ratio formulas for both the phase locked demodulator and the limiter discriminator (15.190) and (15.200) depend on the click/spike rate. Since an approximate theoretical formula for the click rate of a PLL with filter transfer function

\[
P(z) = \frac{1}{2} \int_0^\infty (u - u_0)^2 \left| R(zu) \right|^2 du
\]

is available (14), we will compare the performance of such a phase locked demodulator with that of a limiter discriminator. We rewrite (15.190) in the following way

\[
(SNR)_{15} = \frac{3\mu_0 \left( \frac{\beta}{\gamma} \right)}{\gamma}
\]

where

\[
1 = 24 \text{erf} \left( \frac{\beta}{\gamma} \right) \left( \frac{\beta}{\gamma} \right)
\]

(15.203)

which for a wideband FM reduces to

\[
(SNR)_{15} = \frac{3\mu_0 \left( \frac{\beta}{\gamma} \right)}{\gamma}
\]

(15.208)
where \( \rho = AP_NN_B \)

and (14) (for an unmodulated carrier)

\[
I_A = \frac{4}{\pi} \left( \sqrt{3} - 1 \right) \exp \left( -2 \frac{\beta}{\beta_0} \right) \quad (15.204)
\]

Similarly, the signal-to-noise power output of a limiter discriminator for a wideband FM signal is

\[
\text{SNR}_{1,2} = \frac{3\beta^2}{1 + 4 \sqrt{3} \beta \varepsilon \text{erfc} (\sqrt{\beta})} \quad (15.205)
\]

The results of the computation of (15.203) and (15.205) are shown in Fig. 15.18. From the graphs one finds that a PLD with smaller values of \( B_N / B \) has lower value of the threshold CNR than that of a

![Fig. 15.18. Performance of a phase locked demodulator (PLD) and a conventional limiter discriminator (LD).](image-url)
NonLinear Analysis with Noisy Signals 431

limiter discriminator, in view of this, it may be concluded that we can reduce the threshold CNR by reducing the loop bandwidth ad infinitum. The reduction of the loop bandwidth beyond a certain value cannot be done because of the following reason. The loop must have a certain minimum bandwidth in order that the demodulated output to be undistorted.

In calculating the noise performance, i.e., computing the click tone/spike rate, we have not taken into consideration of the following effects, viz. (i) the effect of modulation on the base of lock probability and (ii) the effect of bursting, i.e., possibility of skipping more than one cycle during the period of unlock. Both of these effects increase the number of clicks at the output and as such the slope of the CNR curve below threshold is increased. Note that the effect of bursting becomes increased with the decrease of the pull-in range compared to the hold-in range.

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CHAPTER 16

DIGITAL PHASE LOCKED LOOPS

16.1 Introduction

With the advent of medium scale integration (MSI) and large scale integration (LSI) in digital circuitry and the use of digital computers in communication systems, a need for digital mechanization of the analog phase locked loops, which comprised of analog circuits, was felt. As a result, a few types of digital phase locked loops, ranging from partially digital to all digital structures, have been proposed [1-6]. In the simplest form, it consists of a phase comparator and a voltage controlled oscillator. A digital phase locked loop (DPLL) can be used for synchronization, demodulation, frequency synthesis and so on. It has all the advantages of digital mechanization, such as reduction in size and cost, improvement in reliability, stability, freedom from drift, etc.

In this chapter, we will not consider various forms of digital phase locked loops, instead we will consider two varieties. Interested readers are referred to the list of references at the end of this chapter.

16.2 All Digital Phase Locked Loop

This type of the digital phase locked loop was first proposed by Pasternack and Whalin [3]. The first order all digital phase locked loop (ADPLL) is essentially a combination of logic circuits driven by two external clocks. The circuit diagram of a first order ADPLL is shown in Fig. 16.1. If consists of an EX-OR gate, a transmission gate (comprising of NAND gates) and a counter of stages. The EX-OR gate acts as a phase comparator and provides an output
Fig. 16.1. The block diagram of an all digital phase locked loop.

gating function depending upon the phase relationship between the incoming signal and the feedback signal. The feedback signal is generated by the counter of $M$ stages ($M = 2^n - 1$) and its half-period is determined by the time required to count $M$ clock pulses, obtained from the output of the transmission gate ($T$-gate). The $T$-gate is driven by two stable clock generators of different frequencies $f$ and $g$ ($f > g$) and is functioning in such a manner as to enable the high frequency clock ($f$) during the '1' state and the low frequency clock ($g$) during the '0' state of the EX-OR output.

In the free running mode, when the input signal is absent, the output of the EX-OR gate is similar to that of the counter. As soon as the gating function, i.e., the EX-OR output, changes to '1' level, the high frequency clock pulses will be transmitted by the $T$-gate to the input of the counter for a time duration, determined by the time required to count $M$ number of pulses. Therefore, the time-span for the level '1' is $Mg$. As soon as the counter changes its state from '1' to '0' after $M$ clock pulses are counted, the $T$-gate will then pass only the low frequency clock pulses for a time period of $Mg$. This is the time required to count $M$ pulses from the clock generator of frequency $g$. In this way the loop repeats the above operation. Hence, the loop oscillator with unequal off- and on-periods. That is, the free-running repetition frequency is $fg/(f + g)M$. In order to derive the equation that governs the behavior of the system, when it is under the influence of an external signal, we refer to Fig. 16.2, showing the input, output and EX-OR output. The principle of deriving the system equation is obviously that the total number of high and low
frequency pulses counted by the counter must equal \( N \).

\[ P(0) \quad P(0+1) \quad P(0+2) \]

\[ Q(0) \quad Q(0+1) \quad Q(0+2) \]

\[ T(0) \quad T(0+1) \quad T(0+2) \]

Fig. 16.2. Illustrating operation of an all digital phase locked loop.

Let the successive on- and off-half-periods of the input and the feedback signals are respectively \( P(k), P(k+1), P(k+2), \ldots \) and \( Q(k), Q(k+1), Q(k+2), \ldots \). The corresponding on- and off-periods of the EX-OR outputs are respectively, \( T(k), T(k+1), \ldots \) and \( P(k) - T(k), P(k+1) - T(k+1), \ldots \). By virtue of the property of \( T \)-gate, the high frequency pulses are transmitted during \( T(k), T(k+1), \ldots \), whereas \( P(k) - T(k), P(k+1) - T(k+1), \ldots \) give the time allocations for enabling the low frequency clock pulses. Therefore, the system equation is given by:

\[ [P(k) - T(k)] g + T(k + 1) f = M \]

i.e.,

\[ T(k + 1) f - T(k) g = M - P(k) g \]  \hspace{1cm} (16.1)

This is a first order difference equation relating the response \( T(k) \) with the excitation \( P(k) \).

If the input signal is a square wave of half-period \( P(k) \rightarrow P(k+1) \) = \( P \), the loop will synchronize provided

\[ T(k) \sim T(k + 1) - T \]  \hspace{1cm} (16.2)

Hence from (16.1) and (16.2), one finds that

\[ T = \frac{M - Pg}{f - g} \]  \hspace{1cm} (16.3)

To find the frequency range of the input signal over which the system remains locked, we note that (cf. Fig. 16.2).
Minimum value of $T = 0$ \hspace{1cm} (16.4)

and

Maximum value of $T = P_{\text{max}}$ \hspace{1cm} (16.5)

The value of the half-period of the input signal, corresponding to (16.4), is given by \hspace{1cm} (16.3)

$$P_{\text{max}} = \frac{M}{\delta}$$

Similarly, the value of the half-period, corresponding to (16.5), is given by \hspace{1cm} (15.3)

$$P_{\text{max}} = \frac{M}{\delta}$$

The lower value of the input frequency for locking to appear is (cf. \hspace{1cm} (16.6)

$$f_L = \frac{\delta}{2\delta}$$

Similarly, the upper value of the input frequency for locking to disappear is given \hspace{1cm} (16.7)

$$f_U = \frac{f}{2\delta}$$

Hence, the locking range (LR) of the first order ADPLL to a harmonic frequency signal is given by \hspace{1cm} (16.9)

$$LR = (f - g)2\delta$$

Note that the locking range on the lower side of the free-running frequency is given by \hspace{1cm} (16.10)

$$LR_L = \frac{f_L}{M(f + g)} - \frac{g}{2\delta} = \frac{f - g}{2\delta}$$

Similarly, the locking range on the upper side of the free-running frequency is given by \hspace{1cm} (16.12)

$$LR_U = \frac{f_U}{M(f + g)} - \frac{f}{2\delta} = \frac{f - g}{2\delta}$$

The relations (16.11) and (16.12) indicate that the upper side lock range is greater than the lower side one ($f > g$).
16.3 Response of a First Order ADPLL to a Frequency Step Input

Assume that for \( t < 0 \), the incoming signal is a \( \Omega' \) case of frequency \( f_k = \frac{1}{T_k} \), lying within the bounds of the locking range. Let us further assume that at time \( t = 0 \) the input signal frequency suddenly changes to \( f_k \), which also lies within the locking range. In such a case, we rewrite (16.1) in the following way:

\[ v(k + 1) - v(k) = M(f_k - G) \]  \hspace{1cm} (16.13)

where

\[ \begin{align*}
\frac{v(k + 1)}{P(k) + G} &= \frac{v(k + f)}{P(k)} \\
F(k) &= \frac{1}{P(k)}
\end{align*} \]  \hspace{1cm} (16.14)

and

\[ G = g G_f \]  \hspace{1cm} (16.16)

Taking the z-transform [7] of (16.13) we write

\[ v(z) - G = (Mf_k - G) \frac{z^{-1}}{z - 1} v(0) \]  \hspace{1cm} (16.17)

where \( v(0) \) is the initial error signal for \( t = 0 \), the normalized input frequency is \( f_k \left( \frac{1}{P_k} = \frac{Mf_k}{f} \right) \)

\[ v(0) = \frac{Mf_k - G}{f} \]  \hspace{1cm} (16.18)

Using (16.17) and (16.18) one gets \( \left( f_k \frac{1}{P_k} = \frac{2G}{f} \right) \) for \( t > 0 \)

\[ v(z) = \frac{Mf_k - G}{1 - G} \frac{z^{-1}}{z - 1} \frac{Mf_k - f_k}{1 - G} \frac{z^{-1}}{z - 1} G \]  \hspace{1cm} (16.19)

which, on taking inverse z-transform, yields

\[ v(k) = \frac{Mf_k - G}{1 - G} - \frac{Mf_k - f_k}{1 - G} G \]  \hspace{1cm} (16.20)

The response, to a frequency step corresponding to (16.30) is seen in Fig. 16.3.

Assuming that \( v(k) \) is a continuous function of time and letting \( v(k) = v(t) \), \( t = kT_k = k/T_f \), we get from (16.20)
Fig. 16.3. Transient response of a first order all digital phase locked loop to a frequency step signal.

\[ M = 120, f_p = 556.4 \text{KHz}, \Delta f = 156.0 \text{KHz}, f_s = 1.1 \text{KHz} \]
and \( f_i = 1.3 \text{KHz} \).

\[ h(t) = \frac{2Mf_s - \Delta f}{f_p - \Delta f} - \frac{2M(f_s - f_i)}{f_p - \Delta f} \exp \left[-\frac{2}{f_s} \ln \left(\frac{f_s}{f_p}\right) \right] \]  
(16.21)

[use the relation \( x = \exp (\ln x) \)].

Therefore, the time constant is

\[ T = \frac{1}{\ln \left(\frac{f_p}{f_s}\right)} \]  
(16.22)

and the corresponding half-power bandwidth is

\[ f_s = \frac{1}{2} f_s \ln \left(\frac{f_s}{f_p}\right) \]  
(16.23)

It is to be noted that the filter shaping depends on the input frequency \( f_s \). It is further to be noted that since the clock signals are not synchronized, a random phase discontinuity results at the time of gating. This results in fluctuation in the output. For further details, interested readers are referred to the work of Pasternack and Wahlin [3].
16.4 Digital Phase Locked Loop

Figure 16.4 shows the block diagramatic representation of a digital phase locked loop. It consists of a sampler, a quantizer, a digital filter and a digital clock. Operational characteristics of the different loop components are described below.

**SAMPLES AND QUANTIZATION**

The sampler works as an electronic watching device. It allows the input signal to pass for a very short duration when it is signaled by the time ticks from the digital controlled oscillator. The output of the sampler obtained at the sampling instants are then quantized to the pre-assigned adjacent discrete level available at the quantizer and are stored as equivalent binary words up to the next sampling instants. Let the input to the digital PLL be represented by

\[ S(t) = A \sin(\omega_0 t + \theta(t)) \]  

where

\[ 0(t) = (a - a_0)\theta + \beta(t) + \theta_0 \]  

Note that \( \omega_0 \) denotes the nominal radian frequency of the digital controlled oscillator (DCO), \( \theta(t) \) is the input angle magnitude and \( \beta(t) \) is the initial phase of the DCO. Obviously, \( \theta(t) \) denotes the open loop frequency error.

The input is sampled at its positive going zero crossing in the sampler port by the time ticks provided by the DCO once in each
period so that the loop is locked onto these zero crossing. The sampler output at the k-th sampling instant is written

$$X(k) = A \sin \left[ \omega_0 (k) + \theta(k) \right]$$  \hspace{1cm} (16.26)

where \( \theta(k) = \theta(t(k)) \) denotes the value of \( \theta(t) \) at the k-th sampling instant.

**Digital Filter**

The digital filter operates on the output of the quantizer and its purpose is to smooth the error signal from the quantizer via the sampler. For a (n+1)st order DLL, the digital filter \( D(z) \) has the form

$$D(z) = \sum_{m=0}^{n} \lambda_m (1 - z^{-1})^m$$  \hspace{1cm} (16.27)

where \( z^{-1} \) denotes the delay operator defined as (see Appendix 16.1)

$$z^{-1}x(k+1) = x(k)$$  \hspace{1cm} (16.28)

and \( \lambda_m \) are constants of the filter.

**Digital Controlled Oscillator**

The DCO in a digital PLL serves the similar purpose to that of a VCO in an analog PLL. Here, the time period is controlled by the digital output, obeying the following algorithm

$$T(f) = T - \gamma (j - 1)$$  \hspace{1cm} (16.29)

where

$$T(f)$$ clock period at the j-th sampling instant

$$\gamma (j - 1)$$ correction signal, i.e., output of the digital filter at \((j - 1)\)th instant.

and

\( T \) Nominal clock period, i.e., the period of the DCO

The relation (16.29) is written because the output of the digital filter is used to correct the phase of the clock by changing the next clock period in such a way as to decrease the phase error.
16.5 Derivation of the System Equation

Figure 16.5 illustrates the operation of the loop when \( \delta(t) \) is a constant phase offset. The X-marks show the instants at which the input is sampled. The sampler output at \( k \)-th instant controls the period of the oscillation at the \( (k - 1) \)th instant. Referring to Fig. 16.5.

![Diagram illustrating sampling of the incoming signal.]

\[ z(0) = 0 \]
\[ t(k - 1) = r(k - 2) + T - y(k - 2) \]
\[ t(k) = r(k - 1) + T - y(k - 1) \] \hspace{1cm} (16.30)

starting from the last expression and substituting upward, we find that

\[ t(k) = t_s + \sum_{i=0}^{k-1} \frac{k-1}{k-1} y(i) \] \hspace{1cm} (16.31)

Taking \( t_s \) to be zero and putting the value of \( t(k) \) in (16.28) one can write the sampler output as

\[ x(k) = A \sin (\theta(k) - \omega_T \sum_{i=0}^{k-1} z(i)) \] \hspace{1cm} (16.32)

where use is made of the relation

\[ \omega_T T = 2\pi \] \hspace{1cm} (16.33)
442 Phase Lock Theories and Applications

Define the phase error as

$$\eta(k) = \theta(k) - \omega_0 \sum_{i=0}^{k-1} y(i) \quad (16.34)$$

Thus the sample output is given by

$$x(k) = A \sin \eta(k) \quad (16.35)$$

Note that the output phase of the DCO at the k-th instant can be written as ($\phi(0) = 0$)

$$\Psi(k) = \omega(1) T(1)$$

$$\Psi(k) = \Psi(k-1) + \omega(k) T(k)$$

starting from the last expression and substituting upwards, we find that

$$\Psi(k) = \sum_{i=1}^{k} \omega(i) T(i)$$

(16.36)

Again $\Psi(k)$ can also be written as

$$\Psi(k) = \omega_0 \eta(k) + \eta(k) \quad (16.37)$$

Again if we refer to Fig. 16.5 and take $\tau(0) = 0$, then it is seen that

$$\tau(1) = \tau(1) + \tau(2)$$

i.e.,

$$\eta(1) = \omega_0 \tau(1) \quad (16.38)$$

Using (16.36), (16.37) and (16.35), we find that

$$h(k) = \sum_{i=1}^{k} \left( \omega(i) - \omega_0 \right) T(i)$$

(16.39)

i.e.,

$$h(k) = \omega_0 \sum_{i=1}^{k} \left( T(i) - \bar{T}(i) \right)$$

(16.40)

Comparing (16.29) and (16.40) we get

$$\eta_0(k) = \omega_0 \sum_{i=1}^{k} y(i) (i-1) = \omega_0 \sum_{i=1}^{k} y(i) \quad (16.41)$$
Here, the phase difference between the input and the output oscillations, 
\[ \varphi(k) = \alpha_k f(k) - \Theta(k) - \omega_0 f(k) - \beta_0(k) \]
\[ = \Theta(k) - \beta_0(k) \]
which on using (16.41) can be written as
\[ \varphi(k) = \Theta(k) - \omega_0 \sum_{j=0}^{k-1} \Theta(j) \]  \hspace{1cm} (16.42)
which justifies the definition of phase difference \( \beta_0 \) as used in (16.39).

Now using the filter network of Fig. 16.6, we write

\[ y(k) = \lambda_1 x(k) + \lambda_2 \sum_{j=0}^{k-1} x(j) \]  \hspace{1cm} (16.43)

From (16.42) and (16.43) one gets
\[ \varphi(k) = \Theta(k) - \omega_0 \sum_{j=0}^{k-1} [\lambda_1 x(j) + \lambda_2 \sum_{j=0}^{j-1} x(j)] \] \hspace{1cm} (16.44)
and
\[ \varphi(k+1) = \Theta(k+1) - \omega_0 \sum_{j=0}^{k} [\lambda_1 x(j) + \lambda_2 \sum_{j=0}^{j-1} x(j)] \]
\hspace{1cm} (16.45)
Therefore, from (16.44) and (16.45) one finds that
\[ \varphi(k+1) - \varphi(k) = \Theta(k + 1) - \Theta(k) - \omega_0 \lambda_1 x(k) - \omega_0 \sum_{j=0}^{k} x(j) \] \hspace{1cm} (16.46)
using the value of \( x(k) \) of (16.33) we get finally
\[ \varphi(k+1) - \varphi(k) = \Theta(k + 1) - \Theta(k) - \omega_0 \lambda_1 x(k) + \lambda_2 \sin \varphi(k) \]
\[ - \omega_0 \sum_{j=0}^{k} \sin \varphi(j) \] \hspace{1cm} (16.47)
which can also be written

\[ \varphi(k + 1) - 2 \varphi(k) + \varphi(k - 1) = 0(k + 1) - 20(k) + 0(k - 1) \]

Equation (16.47) and (16.48) are the system equations of a second order digital phase-locked loop.

In the following we will consider the phase-step and the frequency-step response of the first and second order digital phase-locked loops.

16.6 First Order DPLL with Phase-step Input

For a first order loop, \( \lambda_2 = 0 \), we get from (16.47),

\[ \varphi(k + 1) - \varphi(k) = 0(k + 1) - 0(k) - K \sin \varphi(k) \]

(16.49)

where

\[ K = \omega A \]

Again for a phase-step input,

\[ \theta(k + 1) = \theta(k) \]

hence (16.49) takes the form

\[ \varphi(k + 1) = \varphi(k) = K \sin \varphi(k) \]

(16.51)

If \( \varphi(k) \) is small we write (16.51) as

\[ \varphi(k + 1) = (1 - K) \varphi(k) \]

(16.52)

Taking z-transform of (16.52) one gets

\[ q(z) = \varphi(0) \frac{z}{1 - K} \]

(16.53)

i.e.,

\[ \varphi(k) = \varphi(0)(1 - K)^k \]

(16.54)

From the relation (16.54) it is seen that the loop will attain steady state when

\[ \left| 1 - K \right| < 1 \]

(16.55)

and will not reach steady state, when

\[ \left| 1 - K \right| > 1 \]

(16.56)

\( K_0 = 1 \) may then be considered to be the optimum value of \( K \), because in this case most rapid convergence to the steady state occurs. Responses of the first order DPLL to a phase step input is shown in Fig 16.7, it is seen that for \( \left| 1 - K \right| < 1 \), \( \varphi(k) \) does
Fig. 16.7 Transient response of a first order digital phase-locked loop to a phase-step signal.

Approach zero as \( k \) tends to infinity, otherwise it oscillates between two values. Thus let the corresponding phase error be \( \theta_0 \) and \( \theta_1 \)...

Thus from (16.31), we get

\[
\theta_1 = \theta_0 - K_1 \sin \theta_1
\]

From these relations, it is easily shown that

\[
\theta_1 = \frac{\theta_0}{K_1}
\]

and as such we get from (16.57)

\[
\sin \frac{\theta_0}{K_1} = \frac{2}{\theta_1}
\]
Thus the oscillatory solution and hence instability appears for
\[ K_L > 2.0 \]  
(16.59)

It is worth mentioning that this sort of thing does not appear in
analog phase locked loop.

subsection{16.7 First Order DPLL with Frequency Step or Phase Ramp Signal}
Assuming that the input is a CWS signal with a frequency \( \omega_i \), which is
different from \( \omega_0 \), the nominal frequency. Thus from (16.25) we
write,
\[ \theta(k) = (\omega - \omega_0) \lambda(k) + \theta_0 \]
and
\[ \theta(k + 1) = (\omega - \omega_0) \lambda(k + 1) + \theta_0 \]
i.e.,
\[ \theta(k + 1) - \theta(k) = \frac{d}{dt} \theta(k) \]
(16.60)

Thus from (16.60) and (16.33), we get
\[ \theta(k + 1) - \theta(k) = (\omega - \omega_0) T - (\omega - \omega_0) \int_{t_0}^{t} \frac{d}{dt} \theta(t) \, dt \]
(16.61)

For a first order loop
\[ \lambda_0 = 0 \]
and hence
\[ \phi(k) = \lambda_0 \phi(k) = \lambda_0 \sin \phi(k) \]  
(16.62)

Therefore, from (16.61), (16.62) and (16.49) we get:
\[ \lambda_0 (k + 1) - \lambda_0 (k) = (\omega - \omega_0) T - \omega_0 \lambda_0 \sin \phi(k) \]  
(16.63)

Now we assume that the frequency difference \((\omega - \omega_0)\) is such
that the loop attains steady state. Hence,
\[ \phi(k + 1) = \phi(k) = \phi_0 \]
and we get from (16.62)
\[ \sin \phi_0 = \frac{(\omega - \omega_0)}{\omega_0} \]  
(16.64)

Since the maximum value of \( \sin \phi_0 = \frac{\pi}{2} \) corresponding to \( \phi_0 = \pm \frac{\pi}{2} \), one gets from (16.64)
\[ \pm 1 = \frac{(\omega - \omega_0)}{\omega_0} \]  
(16.65)
Therefore, the upper and the lower locking points, corresponding to $\gamma_u = \pi/2$ and $\gamma_l = -\pi/2$, are respectively given by

$$\gamma_u = \frac{2\pi\omega_0}{2n - \lambda_\Delta A\omega_0} \quad (16.65)$$

and

$$\gamma_l = \frac{2\pi\omega_0}{2n + \lambda_\Delta A\omega_0} \quad (16.66)$$

i.e.,

$$\Omega_u = \gamma_u - \omega_0 = \frac{\lambda_\Delta A\omega_0}{2n - \lambda_\Delta A\omega_0}$$

$$\Omega_l = \gamma_l - \omega_0 = \frac{\lambda_\Delta A\omega_0}{2n + \lambda_\Delta A\omega_0}$$

This indicates that the upper side lock range is greater than the lower side lock range. The total lock range of the DPLL is given by

$$\gamma_u - \gamma_l = \frac{2\pi\omega_0}{2n - \lambda_\Delta A\omega_0}$$

(16.68)

where

$$K_\lambda = \lambda_\Delta A\omega_0$$

Thus the limit of the input distorting $\Omega$ from the centre frequency $\omega_0$ up to which the loop can attain lock is given by

$$\frac{\Omega}{\omega_0} < \frac{\gamma_u - \gamma_l}{2n} \quad (16.69)$$

Since the loop becomes unstable for values of $K_\lambda$ greater than 2, the bounds of the upper and lower side lock ranges are given by

$$-0.244 < \frac{\Omega}{\omega_0} < 0.4669$$

(16.70)

It has been shown [8] that the above condition of locking does not exclude the possibility of cycle slipping or limit cycle operation, which depends on the initial condition as well as on the model gain $K_\lambda$. This is shown graphically in Fig. 16.8. It has been shown [8] that the model gain ($K_\lambda$) needs to be less than 1.15 in order to avoid cycle slipping or limit cycle operation irrespective of the initial condition. As a result, the locking range is given by

$$-0.15 < \frac{\Omega}{\omega_0} < 0.22$$

(16.71)

Corresponding to the value of $K_\lambda = 1.15$. 
If the phase error is not large, we can replace $\sin(q(t))$ by $q(t)$. Thus the response of the first order loop to a step change in frequency can be studied with the help of the $z$-transformation method.

Suppose at time $t < 0$, the frequency of the CW input signal is $\nu_0$, which lies within the lock range. Thus the initial phase error for the linearized loop is

$$q(0) = \left( \frac{\nu_0 - \nu_0'}{\nu_0' \nu_0} \right) 2\pi$$  \hspace{1cm} (16.72)
Now at time $t > 0$, let the input frequency jumps to $o_a > o_b$ lying well within the locking range, consequently the $z$-transform version of (16.65) is given by

$$Z(q) - y(0) z^{-2} = (a_z - a_x + a_x) T_z = a_x \Delta z q(z)\]$$

i.e.,

$$(z - 1 + a_x \Delta z) q(z) = T(a_z - a_x) + y(0)$$

Putting

$$1 - a_x \Delta z = x,$$

we get

$$(z - 1) q(z) = T(a_z - a_x) + y(0)$$

$$q(z) = T(a_z - a_x)\]$$

$$- x T(a_z - a_x)\]$$

$$- 2 x T(a_z - a_x)\]$$

Or taking inverse $z$-transform, this yields

$$q(k) = 2 x T(a_z - a_x)\]$$

16.8 Response of the Second Order Loop to a Frequency Step Signal

The system equation in this case can be written from (16.48) and the relations

$$0(k) = (a - a_o) T(k) + 0_o$$

$$0(k + 1) = (a - a_o) T(k + 1) + 0_o$$

$$0(k - 1) = (a - a_o) T(k - 1) + 0_o$$

i.e.,

$$0(k + 1) - 20(k) + 0(k - 1) = (a - a_o) (T(k + 1) - 2T(k) + T(k - 1))$$

(16.75)
Again from (16.31), we have

\[ t(k) = t_s + kT - \frac{\gamma}{1 + \frac{1}{\lambda} \gamma} \]

\[ 0(k + 1) - 2 \omega(k) + \omega(k - 1) = (\omega - \omega_0) \left[ 2 \frac{\omega}{\omega_0} \gamma + \frac{\gamma}{\omega_0} \gamma \right] \]

\[ = (\omega - \omega_0) \left[ 2 \gamma \omega(k - 1) - \gamma(k - 1) \right] \]

\[ = (\omega - \omega_0) \left[ 2 \gamma (\omega(k - 1) - \omega(k - 1)) \right] \]

(16.76)

Again from (16.43) we get:

\[ y(k) - y(k - 1) = \lambda_X y(k) - \lambda_X y(k - 1) + \lambda \Delta X(k) \]  

(16.77)

Therefore, from (16.77), (16.76) and (16.48) we get:

\[ y(k + 1) - 2y(k) + y(k - 1) = -\omega(\omega - \omega_0)(\omega + \omega_0) \sin \phi(k - 1) \]

\[ = \omega(\omega - \omega_0) \lambda \omega \sin \phi(k) + \omega \lambda \omega \sin \phi(k - 1) \]

(16.79)

Now putting:

\[ \kappa_1 = \omega_0 \lambda \omega \]

\[ \kappa_2 = \omega_0 \lambda \omega \]

we get:

\[ y(k + 1) - 2y(k) + y(k - 1) = \frac{\kappa_1}{\omega_0} \sin \phi(k - 1) \]

(16.79a)

In the steady state:

\[ \phi(k + 1) = \phi(k - 1) \]

That is,

\[ \frac{1}{1 - \frac{\kappa_1}{\omega_0}} \sin \phi \sin \phi = 0 \]

or

\[ \phi = 0 \]

(16.80)
Thus, a second order DPLL, as expected, locks to a frequency step signal with zero steady state phase error. At this point the following question comes up spontaneously for consideration: Is there any limit to the size of the frequency step for locking to occur? Eqn. (16.80) does not indicate the limit of the size of the frequency step, but it can be ascertained from the stability condition of the loop.

This is done in the following way.

Let us try to find the limiting value of \( \delta \) for stable operation of the second order DPLL. We follow the same procedure as outlined in section 16.5. Assume an oscillatory steady state operation and let the corresponding phase error values be \( \gamma_1, \gamma_2, \gamma_3, \ldots \). Hence it follows from (16.79)

\[
\begin{align*}
2(\gamma_1 - \gamma_2) &= \frac{\alpha_2}{\alpha_0} K_e \sin \gamma_1 - \frac{\alpha_0}{\alpha_2} K_r \sin \gamma_2 \\
2(\gamma_1 - \gamma_3) &= \frac{\alpha_2}{\alpha_0} K_e \sin \gamma_1 - \frac{\alpha_0}{\alpha_2} K_r \sin \gamma_3 \\
&\text{etc.}
\end{align*}
\]

i.e.,

\[
\frac{\alpha_2}{\alpha_0} K_e (1 - \tau) \sin \gamma_1 = 0
\]

or

\[
\gamma_1 = -\gamma_2
\]

Hence from (16.81) to (16.83), we get

\[
\sin \gamma_1 = \frac{\alpha_2}{\alpha_0} K_e (1 - \tau) \sin \gamma_1
\]

Therefore, for the second order DPLL to be stable,

\[
\alpha_0 K_e (1 - \tau) = \frac{\alpha_2 K_e}{\alpha_0} (1 - \tau) = \frac{\alpha_2}{\alpha_0} K_e = \frac{4}{\lambda_0} \delta
\]

(16.85)

Together with this let us consider that for sufficiently large values of \( \delta \), the phase error becomes sufficiently small, such that \( \sin \gamma_1 \approx \gamma_1 \).

Hence (16.79) reduces to

\[
\gamma(\delta + 1) \equiv \left( 2 - \frac{\alpha_2}{\alpha_0} K_r \right) \delta \equiv \left( 1 - \frac{\alpha_0}{\alpha_2} K_r \right)
\]

Now let us choose \( \lambda_0 \) and \( \lambda_1 \) in such a way, for the given frequency offset \( \omega_0 - \omega_n \), the following relations are satisfied

\[
\frac{\alpha_0}{\alpha_2} K_r = 2 \quad \text{and} \quad \frac{\alpha_2}{\alpha_0} K_r = 1
\]
Phase Lock Theory and Applications

That is

\[ r = 2, \quad \lambda = \lambda_2 \]  \hspace{1cm} (16.83a)

In this case, it is clear that the second-order phase error \( \eta(k+1) \) will be identically zero, giving "optimum" behaviour of the loop. We find from (16.45) and (16.54a) that when the input frequency is less than the nominal frequency \( \omega_0 \), the loop remains stable irrespective of the value of \( a \) and looks to the input signal with zero steady state phase error, provided \( K_e < 2/3 \). However, when the input frequency is larger than \( \omega_0 \), the stability of the loop is restricted by (16.55). If the size of the frequency step is \( \Omega \) from \( \omega_0 \), then the loop will remain stable provided

\[ \Omega \omega_0 < \frac{4K_e}{(1 + r)K_e} - 1 \]  \hspace{1cm} (16.86)

It is interesting to note that no such condition exists in the case of a second order analog phase-locked loop.

16.9 Response to a Frequency Ramp Signal or a Doppler Rate Input

In this case, we assume that the frequency of the input signal increases linearly with time with a slope \( \alpha \). Moreover, we assume that there is an initial detuning of the amount \( \omega \). Thus \( \Theta(t) \) is written as

\[ \Theta(t) = \Theta_0 + (\omega - \omega_0)t + \frac{\alpha \omega t^2}{2} \]  \hspace{1cm} (16.86a)

Therefore,

\[ \Theta(k+1) - 2\Theta(k) + 3\Theta(k-1) = \lambda_1\alpha (\omega - \omega_0) \sin \Theta(k-1) \]
\[ - (\omega - \omega_0)\lambda_2 \sin \Theta(k) + \alpha \Theta(k) \)
\[ - 2\Theta(k) + \Theta(k-1) \]  \hspace{1cm} (16.87)

Therefore, from (16.45) and (16.87) we get,

\[ \Theta(k+1) - 2\Theta(k) + \Theta(k-1) = -\omega \lambda_3 + \lambda_4 \sin \Theta(k) \]
\[ + \omega_0 \lambda_4 \sin \Theta(k-1) + \frac{\alpha \Theta(k) - 2\Theta(k) + \Theta(k-1)}{2} \]  \hspace{1cm} (16.88)

Again from (16.31) it is easily shown that

\[ \Theta(k+1) - \Theta(k) + \Theta(k-1) = \omega_0 \lambda_4 \sin \Theta(k-1) \]
\[ - \omega \lambda_3 + \lambda_4 \sin \Theta(k) \]  \hspace{1cm} (16.89)
\[ E(\tau) = (u - u_0) + at \] (16.96)

From the discussion of the section (16.7) we know that the loop becomes unstable. When \( t = t_0 \),

\[ u(u_{-} + u_{+}) + u_{+}u_{-} > \frac{4}{1+r} \]

(16.91)

From the relation (16.91) it is seen that after a certain time the system will become unstable, even if it is stable at \( t = 0 \). The range to which lock is easily found from (16.91) by equating

\[ K \left( 1 + \frac{e}{\varepsilon_0} \right) \frac{e}{\varepsilon_0} = \frac{4}{1+r} \]

(16.92)

\[ T_{LL} = \left( \frac{4}{K(1+r)} - \frac{\varepsilon_0}{\varepsilon_0-1} \right) \frac{e_{+}}{e_{-}} \]

16.13 Extended Range DPLL

The block diagram of the extended range DPLL (2RDPLL) \( P \) is shown in Fig. 16.9. It consists of two sampling circuits, the output from each of them is fed to the quantizer, a digital equalizer, a digital lock detector, a digital phase lock detector, and a digital filter. The input signal is decomposed into two with a phase shift of \( e^{-j\pi} \) between them and are fed to the sampling ports 1 and 2. The sampling pulses for the sampling ports 1 and 2 are respectively derived from the leading and trailing edges of the square wave output of the DCO by the use of monostable units. The outputs of the sampler are then amplified to give the error signal in binary words that control the period of the DCO. It has been shown by J. Majumdar (9.13) that the loop
remain stable for $K_i < 4$, whereas for an ordinary DPLL, the condition is $F_i < 2$.

16.11 DPLL Response to Noise Signal

In this section we examine the response of the DPLL when the incoming signal is contaminated with additive white Gaussian noise. Thus the input signal to the DPLL can be written as (cf. Chapter 5),

$$R(t) = A \sin (\omega t + \theta(t)) + K_i(t) \cos (\omega t + \theta(t)) - N(t) \sin (\omega t + \theta(t)) \quad (16.33)$$

where $N_1(t)$ and $N_2(t)$ are independent zero mean Gaussian processes having the same spectral density as that of the input noise but centered around zero frequency.

Therefore, the sample output at the $k$th sampling instant is

$$X(k) = A \sin [\omega k + \theta(k)] + n(k) \quad (16.34)$$

where

$$\begin{align*}
\pi(k) &= N_1(k) \cos [\omega k + \theta(k)] \\
\kappa(k) &= N_2(k) \sin [\omega k + \theta(k)] \quad (16.35)
\end{align*}$$

and

$$n(k) = N_1(k) + N_2(k)$$

Now following the procedure of section 16.5 and noting that

$$X(k) = A \sin \pi(k) + n(k)$$


It is easily shown that for the second order loop, the governing phase equation can be written as for a first order loop

$$\phi(k + 1) - \phi(k) = (h(k + 1) - h(k) - u(k + 1) \sin \phi(k) + n(k))$$

(16.90)

and for a second order loop

$$\phi(k + 1) - 2\phi(k) + \phi(k - 1) = h(k + 1) + 2h(k) + h(k - 1)
- u[k + 1] \sin \phi(k + 1) + u[k] \sin \phi(k - 1)
- \sin \phi(k + 1) + \sin \phi(k) + \sin \phi(k - 1)$$

(16.97)

Before proceeding further with the analysis, it is worthwhile to note that it is difficult to define the spectral density of \( \phi(k) \) explicitly. The presence of the noise makes the sampling interval a non-stationary random variable. The consequence of this non-linearity becomes prohibitively dependent [10, 11] of the loop digital filter. This signal and noise cannot be separated even if the periodic non-linearity (sin \( \phi \)) is ignored.

16.11.1. Linear Analysis: First Order Loop

If the noise term is not high and the phase step signal is considered, then replacing \( x \) by \( S \), we get from (16.96) for a first order loop

$$\phi(k + 1) = (1 - u(k + 1)) \phi(k) - u(k + 1) n(k)$$

(16.98)

Assuming that the noise samples are independent and \( n(k) \) is Gaussian, it is seen that each succeeding \( \phi(k) \) will also be Gaussian. That referring to (16.98) it is seen that as \( k \to \infty \), the mean of \( \phi(k) \) approaches zero and the variance of \( \phi(k) \) is given by

$$\sigma^2 \approx \frac{1}{2} (1 - \alpha \beta) \approx \frac{1}{2}$$

(16.99)

In deriving the above relation it has been assumed that \( \phi(k) \) and \( n(k) \) are independent. We have also assumed that \( n(k) \) is stationary with variance \( \sigma^2 \). In the steady state

$$\sigma^2 \approx \frac{1}{2}$$

and hence from (16.99) we get

$$\sigma^2 \approx \frac{1}{2} \left( 1 - \frac{1}{2} \right)$$

(16.100)

The steady state pdf for the linearized loop is obtained as

$$P(\phi) \approx \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left( -\frac{\phi^2}{2 \sigma^2} \right)$$

(16.101)
From (16.100) it is seen that since the variance \( \sigma^2 \) cannot be negative,

\[
(1 - K_0) \sigma^2 \leq 1.0
\]

i.e.,

\[
K_0 \geq 2.0
\]  \hspace{1cm} (16.102)

This indicates that the phase distribution cannot be a Gaussian one if \( K_0 \) becomes greater than \( \sigma^2 \).

16.11.2. LINEAR ANALYSIS: SECOND ORDER LOOP

In this section we consider the behaviour of a second order DPLL for a frequency step input signal contaminated with additive Gaussian noise. Assuming the size of the frequency step is such that (particularly when the frequency of the input signal is higher than that of the DCO) the DCO locates onto the incoming signal with zero steady state phase error, if the strength of the noise is not large, then the succeeding phase error \( \phi(k) \), \( \phi(k + 1) \), etc. will be small. Thus we can replace \( \sin \phi(k) \) and \( \sin (k + 1) \) by their angles \( \phi(k) \) and \( \phi(k + 1) \) respectively. Hence the system equation (16.97) may be written as remembering that

\[
\theta(i) = (\omega - \omega_0) t + \theta_0
\]

i.e.,

\[
\phi(k) = (\omega - \omega_0) (k + \theta_0) \quad (16.103)
\]

\[
\phi(k + 1) - \phi(k) = K_0 \phi(k) + X_0 \phi(k + 1) - r (K_0 \phi(k) + X_0 \phi(k)) \quad (16.104)
\]

where,

\[
r = 1 + \lambda \beta_0
\]

\[
K_0 = \alpha \beta_0
\]

and

\[
K_0' = \alpha
\]

To arrive at a solution of (16.104) we define

\[
q(k) = \phi(k) - \phi(k + 1) \quad (16.105)
\]

Then substituting this in (16.104) one finds

\[
[q(k + 1) - 2q(k) + q(k - 1)] - r [q(k + 1) + q(k)] - 2q(k + 1) + q(k)
\]
\[ K'(h(k+1)) = K'(h(k)) - \eta(k) + \eta(k+1) + \frac{1}{\lambda} \varepsilon(k) \]

Using (16.106) the following vector equations hold:

\[ y_1(k+1) = y_1(k) \]

\[ y_2(k+1) = 2y_2(k) - y_2(k) + K_v[y_1(k) - \eta(k) + \frac{1}{\lambda} \varepsilon(k)] \]

Putting

\[ a = 2 - rK_v \]
\[ b = K_v - 1 \]

and

\[ K_v^2 = Cv^2 \]

we rewrite (16.107) and (16.108) as

\[ y_1(k+1) = y_1(k) \]
\[ y_2(k+1) = ay_2(k) + by_2(k) + Cn(k) \]

Obviously, in the steady state

\[ y_1 = y_1^* \]
\[ y_2 = y_2^* \]

where \( y_1^* \) and \( y_2^* \) denote the variances of \( y_1 \) and \( y_2 \), respectively. Their covariance is obtained by (16.107a) and (16.108a) as

\[ \gamma_{y_1 y_2} = ab^2y_2^* + bby_2^* + C^2 \]

Again from (16.108a)

\[ \gamma_{y_2} = \gamma_{y_2}^* + \sigma_{y_2}^2 + 2ab\gamma_{y_1 y_2} + C^2 \]

using (16.109), (16.110) and (16.111)

\[ \gamma_{y_2} = \gamma_{y_2}^* + \frac{C^2}{(1 + b)[(1 - b)^2 - \sigma^2]} \]

\[ \gamma_{y_1 y_2} = \frac{C\gamma_{y_2}}{(1 + b)[1 - b^2 - \sigma^2]} \]
Therefore, the correlation coefficient \( \rho \), is given,

\[
\rho = \frac{2\sigma_1 \sigma_2}{\sigma_1^2 + \sigma_2^2} \quad (16.114)
\]

Therefore, the joint distribution of \( x_1 \) and \( y_1 \) is the bi-variate Gaussian pdf and is given by

\[
\rho(x_1, y_1) = \frac{1}{\sqrt{2\pi\sigma_1\sigma_2}} \exp\left[-\frac{(x_1^2 - 2\rho x_1 y_1 + \sigma_1^2)(y_1^2)}{2\sigma_1^2}\right] \quad (16.115)
\]

where

\[
F_1^2 = (1 - \rho^2) \frac{\sigma_1^2}{\sigma_2^2}
\]

Referring to (16.105) we find

\[
\sigma_1^2 = \rho \sigma_1 \sigma_2 (1 - \rho^2)
\]

Thus, in the steady state

\[
E_i(0) = 0
\]

\[
\text{var}(\psi) = \frac{\sigma_i^2}{\sigma_1^2 + \sigma_2^2} = \frac{\sigma_i^2}{\sigma_1^2 + \sigma_2^2} \quad (16.116)
\]

Substituting the values of \( \sigma_1^2 = \sigma_1 \) and \( \sigma_2^2 \) from (16.112) and (16.13) we get

\[
\sigma_i^2 = \frac{(1 - \rho) - 2\rho + \sigma_1^2}{(1 + \rho)(1 - \rho) - \sigma_1^2} \quad (16.117)
\]

For stable solution, \( K_i \) must assume a non-negative value, which is true provided

\[
(1 - \rho)(1 + \rho^2) - 2\sigma_1 \sigma_2 > 0
\]

\[
\text{i.e.,} \quad 2(1 - \rho^2) + K_i(^2 - 1) > 0 \quad (16.118)
\]

and

\[
0 < 1 - \rho \quad (16.119)
\]

\[
(1 - \rho)(1 + \rho^2 - 2\sigma_1^2) > 0
\]

since \( \rho + 2\sigma_1 \sigma_2 \) is always greater than unity the relation (16.116) is automatically true. Thus the condition of stability is obtained from (16.119), as

\[
K_i = \frac{4}{1 + \rho} \quad (16.120)
\]

which is similar to that (16.65) required in the noise-free case.

Since \( \psi \) is Gaussian, its distribution is given
\[ p_2(s) = \frac{1}{\sqrt{2\pi h}} \exp \left( -s^2/2h^2 \right) \]  
(16.121)

16.11.3 NONLINEAR ANALYSIS OF THE FIRST ORDER LOOP WITH NOISE SIGNAL

It is clear from the loop equation that \( q(t) \) is a discrete time, continuous variable Markov Process. Thus \( q(t), q(t+1), \ldots \) \( q\ldots \) where \( q(t) \) are Markovian sequence of random variables, and hence the conditional distribution of \( q(t+1) \) given \( q(t) \) only.

That is, 
\[ p(q(t+1) \mid q(t), \ldots, q(0)) = p(q(t+1) \mid q(t)) \]  
(16.122)

\[ p(q(t+1) \mid q(t)) \] is sometimes called transition probability of \( q \). Similarly the joint distribution of \( q(0), q(1), \ldots, q(t+1) \) can be written as

\[ p(q(0), q(1), \ldots, q(t+1)) = \prod_{j=0}^{t} p(q(j+1) \mid q(j)) \]  
(16.123)

Again it is easily shown by the application of theorem of conditional probability that the following holds \([12, 14]\) for a Markov process:

\[ p(q(t+1) \mid q(t)) = \sum_{k=0}^{t} q(k) p(q(t+1) \mid q(t)) \]  
(16.124)

This is the Chapman Kolmogorov equation. This indicates that starting with \( q(0) \) at \( 0 \) one can jump to a value \( q(t) \) at some arbitrary later time \( t \). Then with this new values of \( q(t) \) as a starting point, one can compute the probability of finally being in the range \( (q(t+1), q(t) + q(t+1) + p(q(t+1) \mid q(t)) \) at a later time \( (q(t+1) + 1)(q(t+1) + 1) > q(t) > 0 \). This illustrated in Fig. 16.10.

We rewrite (3) as

\[ p_{n+1}(s \mid \eta) = \int p(q(t+1) \mid q(t), \ldots, q(0)) \]  
(16.125)
where

\[ \psi_0 = \eta(0) \]

\[ p_\psi(\psi | \psi_0) = \text{pdf of } \psi(t) \text{ given } \psi_0 = \psi(0) \]

\[ q_\psi(\psi | x) = \text{transition pdf of } \psi(t+1) \text{ given } \psi(t) = x. \]

Referring to the phase equation, the transition pdf is seen to be Gaussian with mean

\[ E(\psi | x) = \psi - K_\psi \sin x \]  \hspace{1cm} (16.126)

and variance

\[ \text{Variance} = K_\psi^2 \sigma^2 = K_\psi^2 \psi^2 / A^2 = \nu^2 \]  \hspace{1cm} (16.127)

where \( \sigma^2 \) denotes the noise. Thus

\[ q_\psi(\psi | x) = \frac{1}{\sqrt{2\nu^2}} \exp \left[ -\frac{(\psi - \psi + K_\psi \sin x)^2}{2\nu^2} \right] \]  \hspace{1cm} (16.128)
Usually one is interested in the value of $q$ lying within $-\pi$ and $\pi$. Therefore, we rewrite (16.125) as

$$W_{\nu}(q | \nu) = \frac{1}{2\pi} Q_\nu (q | z) W_\nu (z | \nu) \, dz$$  \hspace{1cm} (16.129)

where

$$W_{\nu}(q | z) = \sum_{\nu = \pm} p_\nu (q + 2\pi n | \nu)$$  \hspace{1cm} (16.130)

$$Q_\nu (q | z) = \sum_{\nu = \pm} q_\nu (q + 2\pi n | z)$$  \hspace{1cm} (16.131)

i.e.,

$$Q_\nu (q | z) = \frac{1}{2\pi}, \exp \left(-\frac{r^2 + 2\nu z + z^2}{2\nu^2} \right)$$  \hspace{1cm} (16.132)

$W_{\nu}(q | \nu)$ can be calculated from a knowledge of $W_\nu (q | \nu)$ using (16.129) and (16.132) and taking successively $\nu = 1, 2, \ldots$ with $q_\nu = 0$ and $W_\nu (q | \nu) = \delta (z - q_\nu)$. In the steady state, i.e., $\nu = \infty$, the pdf $W(q)$ is given by

$$W(q) = \int_{-\pi}^{\pi} Q_\nu (\tilde{q} | z) \, \tilde{q} \, d\tilde{q}$$  \hspace{1cm} (16.133)

Now realizing that a DPLL is a nonlinear system with a periodic nonlinear element, it is evident that there will appear threshold effect. In order to predict the existence of the threshold effect it is necessary to compute the phase error variance of the nonlinear loop and compare with the phase error variance of a linearized loop. From the nature of the pdf of $q$, it is easily recognized that it is not possible to analytically compute $\xi_\nu^2$, but it can only be computed numerically via a digital computer. On the other hand, the effect of the nonlinear operation is reflected if one replaces $sin z$ by $z - z^3/6$. Thus with this approximation, we find that the variance of $q$ is given by

$$\xi_\nu^2 = \int_{-\nu}^{\nu} W(q) \, dq$$

$$= \int_{-\nu}^{\nu} W(q) \, dq \int_{-\nu}^{\nu} \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{r^2 + 2\nu z + z^2}{2\nu^2} \right)$$  \hspace{1cm} (16.134)
462 Phase Lock Theories and Applications

If the SNR is high, we assume that \( W(c) \) is Gaussian with variance \( \sigma_w^2 \). Thus

\[
\sigma_w^2 = \sigma_x^2 + (1 - K_1) K_1 \sigma_e^2 + 5K_1^2 \sigma_e^2/12
\]  

(16.135)

Knowing this, we can compute the threshold value of \( \sigma_w^2 \), by defining the threshold point as that value of \( \sigma_w^2 \) for which \( \sigma_e^2 \) differs \( \sigma_e^2 \) by 1.0 dB. That is,

\[
\sigma_w^2 = 1.26 \sigma_e^2
\]  

(16.136)

Hence from (16.135) and (16.136), one gets

\[
(1 - K_1) K_1 \sigma_e^2 + 5K_1^2 \sigma_e^2/12 = 0.26
\]  

(16.137)

From (16.109) and (16.137) one easily gets the threshold value of \( \sigma_0 \),

\[
\sigma_0^2 = \left( \frac{2}{k} \right) - 1.2(2 - K_1) (K_1 - 1) \pm \sqrt{1.45 - (2 - K_1)K_1}
\]  

(6.138)

16.12 Second Order DLL and Filter Design

The purpose of this section is to explore the possibility of using a second order DLL as a second order low pass Butterworth filter. Keeping aside, for the moment, the question of a DLL, let us first consider an Analog Phase Locked Loop, and observe the output after the low pass filter. Then we write the output as

\[
\gamma(s) = \frac{AF(s)}{s + F(s)AK} (\theta(s))
\]  

(16.139)

Now to realize a second order Butterworth response, we take \( F(s) = 1/(s + T) \) and rewrite (16.139) as

\[
\gamma(s) = \frac{AF(s)}{s + AT + AK} (\theta(s))
\]  

(16.140)

If the Butterworth response has a cut off at \( \omega_c \), then it is obvious that

\[
x^0 + s^2T + AKT = s^2 + (2\omega_c)k + \omega_c^2
\]  

(16.141)

That is,

\[
x^0 = AKT \frac{1}{T} = (2\omega_c)^k \omega_c
\]
or. \( u_2 = (2)\sqrt{dK} \)  

Thus the poles of the response function we obtained are (cf. 16.141)

\[
x_1, x_2 = \frac{\Omega}{2} (-1 \pm j) \tag{16.145}
\]

Inserting this \((16.140)\), we can calculate the time response \( V(t) \), provided \( \delta(t) \) is known. For example, let us suppose that the APLL \( \delta \) desired to receive an FSK signal, which swings by \( \pm \Omega \) around the centre frequency of the VCO. Let us suppose that at time \( t = 0 \), the VCO is locked to the signal which is shifted by \(- \Omega\) from the centre frequency. That is, we write

\[
\delta(0) = -\frac{\Omega}{AK}, \quad \delta(t) = 0
\]

Thus

\[
V(t) = A_T \frac{\delta(t)}{\beta^2 + \nu T + AK T} \delta(t) + \frac{\delta(t) + \nu T}{\beta^2 + \nu T + AK T} \delta(0)
\]

which for a frequency step \( \Omega \) at time \( t = 0 \), becomes

\[
V(t) = A_T \left[ 1 \frac{\Omega}{AK T} \right] - 1 \frac{\beta + 1/2}{\beta^2 + \nu T + AK T} \tag{16.144}
\]

Therefore,

\[
V(t) = A_T \left[ \frac{\Omega}{AK T} \right] 2I - 2 \exp (-\frac{\nu T}{2}) \sin \left( \nu T (2)^I \right) + \cos \left( \nu T (2)^I \right) \tag{16.145}
\]

To have a second order Butterworth Response with a DPLL, we consider a DPLL with the following filter

\[
D(x) = \frac{x^2 - \frac{\Omega}{AK T}}{z^2 - \frac{\Omega}{AK T}}
\]

and the output of this filter is used to correct the frequency of the DCO. Thus

\[
T(1) = T - AD \sin \phi (j - 1)
\]

Again

\[
\theta(j) = \frac{\pi}{2} (2\nu(n) - \nu(n)) T(j)
\]
\begin{align}
\psi(j) &= 0(j) - \theta_0(j) \\
\text{The above equation can also be written as} \quad \psi(k + 1) - \psi(k) &= a_0 A D(c) \sin \psi(k) \\
\text{where} \quad \psi(k) &= 0(k) - \theta_0(k) \\
\text{and} \quad \psi(k + 1) - \psi(k) + a_0 A D(c) \sin \psi(k) &= \theta_0(k + 1) - \theta_0(k) \\
\text{Now for a frequency-step input signal} \quad \phi(k) &= (\omega - \omega_0) t(k) + \theta_0 \\
\text{where} \quad t(k) &= kT - \sum_{j=0}^{k-1} A D(c) \sin \psi(j) \\
\text{which on comparison with the expression of } \phi(k) \text{ yields} \quad \psi(k) &= kT - \sum_{j=0}^{k-1} a_0 A D(c) \sin \psi(j) \\
\text{Therefore,} \quad \phi(k) &= (\omega - \omega_0) t(k) - \theta_0(k) + \theta_0 \\
\text{Hence the phase equation is written as} \quad \psi(k + 1) - \psi(k) + a_0 A D(c) \sin \psi(k) \\
\quad &= 2\pi \frac{\omega - \omega_0}{\omega_0} - \theta_0(k + 1) - \theta_0(k) \\
\quad &= 2\pi \frac{\omega - \omega_0}{\omega_0} - \theta_0(k) + a_0 A D(c) \sin \psi(k) \\
\text{i.e.,} \quad \psi(k + 1) - \psi(k) + \frac{\omega}{\omega_0} (a_0 A D(c) \sin \psi(k)) &= 2\pi \frac{\omega - \omega_0}{\omega_0} \\
\text{Now using the expression for } D(c), \text{ we rewrite the linearized phase equation (sin } \psi(k) \approx \phi(k)) \text{ as} 
\end{align}
\[ y(k + 2) + 2y(k + 1) \left(-1 - b + \frac{c_1}{\omega_0} (1 - b)\right) + b_1 y(k) = 2z_0 (1 - a_0) (1 - b) \quad (16.153) \]

Using the method of \( r \)-transform
\[ \zeta = (r + 1) / (r - 1) \]

and putting
\[ b_1 = \frac{c_2}{\omega_0} K_1 - b - 1 \quad (16.154) \]
\[ K_1 = c_0 A_0 \]

it is easily shown that stability requires
\[ 1 + b_1 + b > 0 \]
\[ 1 - b_2 + b > 0 \]
\[ 1 - b > 0 \]

Therefore, to have a Butterworth response, the poles in the \( z \)-plane are given by
\[ z_p = \exp \left( \frac{-2 \pi f_s}{f_0} \right) \]

\[ f_s \text{ being the sampling interval, which may be taken to be the nominal period of the DCO. That is,} \]
\[ z_1, z_2 = \exp \left( \frac{-\pi (a \pm \delta)}{f_0} \right) \]

where
\[ a = \frac{2\pi f_s L_0}{f_0} \]

Hence the characteristic equation with \( z_1 \) and \( z_2 \) is written as
\[ (z - z_1)(z - z_2) = z^2 - (z_1 + z_2) z + z_1 z_2 \]

\[ (16.158) \]

Comparing this with the characteristic equation and putting the values of \( z_1 \) and \( z_2 \) one gets
\[ b_1 = - (z_1 + z_2) = -2 \exp \left( -\pi a \cos \delta \right) \]

and
\[ h = \exp \left( -2\pi a \right) \]

Thus
\[ 1 - b > 0 \text{, since } a \text{ is positive} \]

Again \( 1 + b_1 + b > 0 \) (cf. 16.155).
The charge pump behaves like an ideal integrator and its output becomes positive when the UP line is energized, whereas its output goes negative when the DOWN line is energized. Therefore, the frequency of the VCO will be pulled up or down depending upon whether the frequency of the incoming signal is higher or lower than that of the VCO till the frequency difference is reduced to zero. Obviously, the final phase error will be zero because of the integrating action of the charge pump. The pull-in and hold-in ranges are infinite for the integrating action of the charge pump, i.e., in practice, they
are as large as the free frequency of the VCO. This type of DPLL will find at wider use in frequency synthesizers, which are required to generate a frequency within a large frequency band.

Finally, the DPLLs are not as good as the linear or analog PLLs in respect of their performances in a noisy environment. This happens mainly due to the fact that the linear multiplication process of the input and output signals averages out any components of the input signals not correlated with the VCO output. Whereas in DPLLs, which operate in the saturation mode, this type of cancellation of noise is not possible, because they respond to zero crossings of the input signals.

APPENDIX 16A

Introduction to Z-Transform

In dealing with the problems of linear time-continuous systems, we often make use of the Laplace transform. This simplifies analysis. However, if the Laplace transform method is applied to linear time discrete systems, then the transfer functions of the systems turn out transcendentals in form. For example, let us consider the output of an ideal samples, which samples continuous time function \( f(t) \) with the help of a train of impulses (Dirac function) at discrete sampling instants spaced by constant time interval \( \Delta t \). Denoting the output of the sampler by \( f^*(t) \), we write

\[
\begin{align*}
    f^*(t) &= f(t) \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) \\
    &= \sum_{n=-\infty}^{\infty} f(n\Delta t) \delta(t - n\Delta t)
\end{align*}
\]

Note that the strength of the impulse output (impulse area) are equal to the magnitude of \( f(t) \) corresponding to the sampling instant. Taking the Laplace transform of \( f^*(t) \) we write

\[
F^*(s) = F(s) = \sum_{n=-\infty}^{\infty} f(n\Delta t) \exp(-ns) \tag{2}
\]

The appearance of the transcendental function, \( \exp(-ns) \), complicates the analysis considerably and as such is a new variable \( z \), related to \( s \) through the following relation:

\[
z = \exp(\Delta t s), \quad i.e., s = \frac{1}{\Delta t} \ln z \tag{3}
\]
is used. Note that since \( z \) is a complex variable so also \( x \) is. Thus we write,

\[
s = \tilde{z} + j\tilde{o}
\]

\( (6) \)

and

\[
z = x + jy
\]

\( (5) \)

Now using (3), (4) and (5), we get

\[
z = x + jy = \exp ((\theta + j\omega)\Delta t)
\]

That is,

\[
x = \exp (\theta + j\omega) \sin (\omega \Delta t)
\]

\( (6) \)

\[
y = \exp (\theta + j\omega) \cos (\omega \Delta t)
\]

\( (7) \)

in polar coordinates in the \( z \)-plane i.e., \( r = r \exp (j\theta) \), we write

\[
r = \exp (\theta + j\omega) \Delta t
\]

and

\[
\theta = \omega \Delta t + 2\pi x
\]

The conversion of the \( z \)-plane to \( z \)-plane is shown in Fig. 16.1A.

Using (2) and (3) write

\[
P_z \left( x = \frac{1}{i} \ln z \right) = \sum_{n=0}^{N} f(n\Delta t)x^{-n}
\]

\( (8) \)

This is called the \( z \)-transform of \( f(n) \). Before we proceed to construct a table of \( z \)-transform, we give some transformations of some characteristic points and regions from the \( z \)-plane to the \( z \)-plane. These are shown in Fig. 16.2A.

Referring to the relation (8), it is seen that the point \( z = 0 \) transforms to the point \( z = 1 \) in the \( z \)-plane (cf. Fig. 16.2A). A point
\[ z = -j \frac{\pi}{\Delta t} \text{ corresponds to } z = \text{cis}(-j\pi) = 1. \] A point \[ z = -j \frac{\pi}{\Delta t} \text{ corresponds to } z = j \]. Thus when a point in the \( z \)-plane moves on the \( jo \)-axis from \( jo \) to \( jo \Delta t \), it traces out a circle of unit radius in the anti-clockwise direction in the \( z \)-plane. Obviously, if the point in the \( z \)-plane moves from \(-jo\) to \( jo \), the point in the \( z \)-plane describes an infinite number of circles. Similar explanation may be provided for other examples of Fig. 16.2A.

![Diagram](image)

**Fig. 16.2A.** Illustrating transformations from a \( z \)-plane to a \( z \)-plane.

Now let us come back to the proposition for the formation of \( z \)-transform pairs. We consider the following

**Case 1:** \( f(t) = K \)

\[ k.e., \quad f(s) = K \]
472. Phase Lock Theories and Applications

Hence,

\[ F(x) = \sum_{n=0}^{\infty} Kx^{-n} = K \left(1 + \frac{1}{x} + \frac{1}{x^2} + \ldots \right) \]

\[ = \frac{x}{x - 1} \]

(9)

Case II: \( f(t) = \exp(at) \)
i.e., \( f(\theta) = \exp(an) \), when \( a = b \Delta t, n = t/\Delta t \)

Hence

\[ F(x) = \sum_{n=0}^{\infty} e^{an}x^{-n} = \frac{x}{x - e^{-a}} \]

\[ = \left[1 + \frac{1}{x - e^{-a}} + \frac{1}{x - e^{-a}} + \ldots \right] \]

i.e.,

\[ F(x) = \frac{xe^{a}}{xe^{a} - 1} = \frac{x}{x - e^{a}} \]

(10)

Case III: \( f(t) = \frac{1}{\Delta t} \text{te}^{bt} \)
or

\[ f(t) = \frac{\partial}{\partial t} (\text{e}^{at}) \]

Therefore,

\[ F(x) = \frac{\partial}{\partial x} \left[ \frac{x}{x - e^{a}} \right] \]

\[ = \frac{e^{a}}{(x - e^{a})^2} \]

i.e.,

\[ F(x) = \frac{xe^{a}}{(x - e^{a})^2} \]

(11)

Referring back to the relation

\[ F(x) = \sum_{n=0}^{\infty} f(n\Delta t) x^{-n} \]

we rewrite it, putting a new integer \( k = n - 1 \),

\[ F(x) = \frac{x}{x - e^{a}} f(k + 1) x^{-k+1} \]
\[-\left[ f(k+1)\Delta t \right] z^{-1} + \frac{1}{\Delta t} \sum_{n=0}^{\infty} f(n+1)\Delta t z^{-n}\]

i.e.,
\[\sum_{n=0}^{\infty} f(n+1)\Delta t \cdot z^{-n} = zF(z) - z^2 f(0)\]

Hence
\[\frac{z}{z - \Delta t} F(z) = zF(z) - z^2 f(0)\]

Thus
\[zF(z) - z^2 f(0) = zF(z) - z^2 f(0)\]

Table of \( \nu \)-transforms

<table>
<thead>
<tr>
<th>Time function</th>
<th>( \nu )-transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Unit step ( u(t) )</td>
<td>( \frac{z}{z - 1} )</td>
</tr>
<tr>
<td>2. ( (t - \Delta) )</td>
<td>( \frac{z}{z - 1} )</td>
</tr>
<tr>
<td>3. ( e^{-\Delta t} )</td>
<td>( \frac{z}{z - \Delta} )</td>
</tr>
<tr>
<td>4. ( \sin at )</td>
<td>( \frac{z \sin (a/\Delta)}{z^2 - 2z \cos (a/\Delta) + 1} )</td>
</tr>
<tr>
<td>5. ( \cos at )</td>
<td>( \frac{z \cos (a/\Delta)}{z^2 - 2z \cos (a/\Delta) + 1} )</td>
</tr>
<tr>
<td>6. ( e^{\Delta t} \sin at )</td>
<td>( \frac{z \sin (a/\Delta)}{z^2 - 2z \cos (a/\Delta) + 1} )</td>
</tr>
<tr>
<td>7. ( e^{\Delta t} \cos at )</td>
<td>( \frac{z \cos (a/\Delta)}{z^2 - 2z \cos (a/\Delta) + 1} )</td>
</tr>
</tbody>
</table>
   Nachrichten, Nachr. 6/1975, 8/1972, 10/1975: Signal Processing with Phase-
   Locked Loops, 1975.
Applications of phase lock principles are many and in fact it encompasses various fields of science and technology, such as, physics, biophysics, biology, telecommunication, electrical engineering, etc. As such, it is not possible to incorporate all of them in a book of this nature; instead, we will concentrate on certain applications concerning signal processing. As a matter of fact, some of the applications in this area have already been cited in the earlier chapters. Incidentally, phase lock principles are utilized in signal processing at r.f. as well as higher band of frequencies. At microwave and multimeter wave frequency regions, (1-10) the principle of injection synchronization is commonly used because of the simplicity of circuit arrangement. In these frequency regions, mostly solid state oscillating devices, like Gunn, IMPATT, etc. are used nowadays. A simple arrangement for injection synchronisation is shown in Fig. 17.1. It consists of a Gunn or an IMPATT oscillator, and a circulator. The synchronization characteristic of microwave solid state oscillators are somewhat different from those of the oscillators operating at r.f. frequencies. The differences arise mainly because of two reasons, viz., (1) microwave solid state devices are modelled as nonlinear admittance functions rather than as nonlinear conductance functions, like in a van der Pol oscillator (cf. Chapter 2), and (2) aside their nonlinear dependence on the amplitude of the device current or voltage, they are nonlinear functions of the frequency of the oscillating current or voltage. As a result, the locking characteristics of a synchronized oscillator become asymmetric in nature on the two sides of the centre frequency. This is easily explicated by modifying van der Pol's theory. Anyway the basic properties of injection synchronization
Further Applications of Phase-Lock Principles 477

Fig. 17. Injection synchronization of an Gunn/IMPATT oscillator.

remain the same as those of r.f. oscillators. The locking range, realized in this way, may not be sufficient for certain purposes. In such cases, an auxiliary FM detector can be used to extend the locking range by applying it to a varactor tuned Gunn/IMPATT oscillator. However, one can also utilize this voltage to bias such an oscillator, that is particularly suitable at mm-wave frequencies.

For certain applications, synchronization of microwave IMPATT oscillators by means of modulated laser beams [11-19] has also been proposed recently, because of certain advantages, viz., d.c. isolation, simple interface, immunity to electrical interference, low injection power requirements, etc. A simple arrangement is depicted in Fig. 172. Here a laser beam is modulated by the synchronizing signal and then the beam is directly injected into the active region of the IMPATT diode, constituting the oscillator to be synchronized. The nonlinearity, that is responsible for synchronization in this case, is the nonlinear dependence of the injection rate for the carriers in a semiconductor on the electric field over the active region of the diode. The characteristics of a microwave oscillator synchronized through the optical terminal are different from those of a microwave oscillator synchronized through its electrical terminal. These are, viz., (1) The locking range, here, at first increases with the increase of the strength of the local oscillator output, but when it becomes
high the locking range starts increasing with the oscillator output. (ii) Asymmetry in the locking range is here more pronounced. (iii) The oscillator can be optically tuned by changing the intensity of the laser beam.

17.1 Oscillator Cleaning

Because the various noise sources within the oscillator circuit, there appears a jitter in the output phase of the oscillator output, leading to spectral broadening of the output waveform. As a result, spectral purification of the output waveform becomes necessary for many applications related to generation of secondary time and frequency standards. Further, it is known that the crystal oscillator gives the best long-term stability when operated at a low level. On the other hand, the short-term phase stability is realized at a higher power level operations. Best result is expected when both are combined. This is achieved by applying a pure signal either directly to the oscillator using the principle of injection synchronization or indirectly to the oscillator using the phase lock loop. Because of the simplicity in the circuit arrangement of an injection synchronized
oscillator, the additional sources of noise in a PLL, viz., phase detector noise, filter noise, etc., do not contaminate the output. As a result, injection synchronization is helpful in this respect. PLL techniques are preferred in the field of frequency synthesis [17, 18, 19]. However, in the case of frequency division, where the ratio of frequency is not large, the principle of ultraharmonic synchronization may be utilized. Incidentally, ultraharmonic synchronization is the technique of synchronizing an oscillator with a signal having a frequency which a multiple of that of the oscillator (20–23).

Fig. 17.3. Demodulation techniques for FSK signals using (a) a phase-locked loop, and (b) an injection-synchronized oscillator.
17.3 Demodulation

We have already indicated the use of PLL for the demodulation of an analog angle modulated signal. A PLL can also be used in the demodulation of PSK and FSK signals [24-26]. Schemes for the detection of binary signals are shown in Fig. 17.3 and those for the binary PSK signals are depicted in Fig. 17.4. Fig. 17.3a and Fig. 17.3b respectively depicts the PLL and injection synchronized oscillator (ISO) configurations for the detection of a binary FSK signal. A binary FSK signal may be represented as

\[ v(t) = \sqrt{2} \hat{A} \sin (\omega t + \xi(t)) \]

where

\[ \frac{d\xi}{dt} = + \Delta, \text{ for signal 1} \]

and

\[ \frac{d\xi}{dt} = - \Delta, \text{ for signal 0}. \]

For a PLL that is operating in the locked state, it is easily shown that when the loop gain is high compared to unity, the filter output is proportional to the rate of change of the input phase variation with time, i.e.,

\[ v_d(t) = \frac{1}{K_v} \frac{d\xi}{dt} \]

Hence the output of the filter network gives the detected signal. Fig. 17.3b utilizes a narrowband ISO, which generates the reference carriers. This reference carrier when multiplied with the incoming FSK signals leads to demodulation. Fig. 17.4 shows two schemes for the demodulation of a binary PSK signal. A binary PSK can also be represented as

\[ v(t) = \sqrt{2} \hat{A} \sin (\omega t + \Psi_1(t)) \]

where

\[ \Psi_1 = 0 \text{ for signal 1} \]

\[ \Psi_1 = \pi \text{ for signal 0}. \]

In the case of the squaring loop of Fig. 17.4a, the signal on passing through the squarer looses the \(0\) and \(\pi\) information in the phase. The squarer output \(2A^2 \sin^2 \omega t = A^2 (1 - \cos 2\omega t)\) actsuates a PLL operating at a frequency of \(2\omega_c\). The output of the VCO is
frequency divided by 2. The divider output is used to multiply the input signal in order to produce the detected output. Fig. 17.6b
gives the Costas loop for the detection of a PSK signal. Here the VCO operates at the harmonic frequency (i.e., uo). The output of
the VCO is broken into two signals differing in phase by \( \pi /2 \). These
two outputs are multiplied with the input signal to generate two signals: one is proportional to the sine of the phase difference
between the input and the output and the other is proportional to the cosine of the phase difference. These two signals are again multipli-
ced to remove the information about 0 or \( \pi \) phase shift. The output
gives the control signal for the VCO and quadrature channel gives the detected output.

Figure 17.5 sketches an arrangement for the detection of an AM

![Fig. 17.5. The demodulator for an AM signal.](image)

signal. The input signal \( 2d (1 + mf(t)) \sin(cot + \theta_0) \) is divided into two parts. One of these feeds the limiter in order to remove the amplitude modulation and the other goes to a multiplier. The limiter output feeds a PLL which in turn generates a coherent/semicoherent signal. The VCO output is \( \pi /2 \) phase-shifted to provide the other signal for the multiplier. The output of the multiplier is filtered to produce the detected signal.

17.3 FM/PM Modulation

An FM or PM wave can be generated by directly applying the modulating signal or a differentiated version of the same to the in-
put of a voltage-controlled oscillator. The disadvantage of producing an angle modulated wave in this way is that the carrier frequency
is not derived from a highly stable oscillator. In order to overcome
this difficulty, a PLL can be used [27] in a fashion as shown in Fig.
17.6. The output of the VCO is phase compared with that of a crystal

Fig. 17.6. Generation of an angle modulated wave.

controlled oscillator. The phase detector output is added with the
modulating signal or its differentiated version to control the instant-
aneous frequency of the VCO. Choosing the loop gain to be large
compared to the highest frequency of the modulating signal, it is
easily seen that the instantaneous frequency modulation is given by

$$\frac{d\theta}{dt} = \frac{d\theta}{dt} = K_P \cdot f(t)$$

where $\theta(t)$ is the instantaneous phase modulation of the VCO and
$V_m(t)$ is the modulating signal in case of frequency modulation or
the differentiated modulating signal in the case of phase modulation.
Since in this case VCO output is compared with the output from a
crystal oscillator, the FM/PM transmitter will have the current carrier
frequency. The linearity of the loop and the choice of the filter net-
work in order to realize the desired frequency response characteristics
are the design considerations.

17.4 Filtering

In building communication transmission sub-systems, the need for
filters that are required to keep the transmitted spectrum within a
specified limit, is often felt. For this purpose a linear PLL can be
used with advantages [28]. Remembering that the transfer function
of the loop is

$$H(j\omega) = \frac{A(j\omega)}{1 + j\omega}$$
It is not difficult to realize the desired response function by using cataloged analog filter data for \( F(s) \). In designing such a PLL filter, if the open loop gain varies, questions regarding the stability of the loop arise, particularly when higher order systems are made use of. But, in such applications a system modulator usually operates in a large signal to noise environment, where the open loop gain can be considered constant, and as such stable operation of the loop could be insured. Linear PLL filters have certain advantages over the conventional filters [22] and those are enumerated below:

**PLL Filters**

1. Highly nonlinear 2-port network.
2. In the locked state, input and output frequencies of the 2-port are unequal.
3. In the locked state, input and output frequencies are equal. Output amplitude is independent of the input amplitude.
4. Automatic frequency tracking is a special feature.
5. Coherent signal processing device providing excellent noise rejection.
6. Bandwidth can be made very small, say 10 kHz in 1 GHz, by varying filter time constant.
7. Being a tracking filter, most PLL's do not require high precision components.

**Conventional Filters**

1. Linear 2-port network.
2. Input and output frequencies are equal.
3. Input and output frequencies are equal. Output amplitude is a linear function of the input amplitude.
4. Conventional filters do not provide automatic frequency tracking.
5. It is not a coherent signal processing device.
6. It is not possible.
7. High precision components are required.

**17.5 Colour Identification in Television**

A PLL plays an important role in identifying colours in a colour TV. In colour television systems, suitable combinations of the red (K), blue (B) and green (G) components of a picture are made to produce three signals, which are transmitted. These signals are

\[ Y = 0.30 \times K + 0.59 \times G + 0.11 \times B \]
Further Applications of Phase Lock Principles 435

and

$Q = 0.21 R - 0.52 G + 0.51 B$.

The signal $Y$ is called the luminance signal, whereas $I$ and $Q$ are called chrominance signals. The luminance signal is identical with the conventional monochrome television system and transmitted accordingly. The $I$ and $Q$ signals balance modulate two subcarriers, having the same frequency (say, 3.575 MHz), but differing by a phase offset of 90° degrees. Then $I$, $Q$ and $Y$ channels are modulated to form the net transmitted signal. A colour burst of at least eight cycles of the same subcarrier frequency at a standard phase is also transmitted at a rate of the line frequency (15,625 kHz in NTSC system) for recovering the reference carrier at the receiver.

At the receiver and the $I$ and $Q$ signals are extracted with the help of two synchronous detectors fed with two reference subcarriers separated by a phase difference of 90 degrees. One of the reference subcarriers, which is taken from the VCO output, is generated by feeding the colour subcarrier burst to a PLL [3] and the other is derived from it by shifting its phase by 90 degrees. It is important to note that here the PLL has to lock onto an intermediate signal at a rate of 15.625 KHz. As such it is possible that the loop may lock onto any of the frequencies, offset from the subcarrier frequency by integer multiples of the line frequency, viz., 3.575, 3.6 ± 0.015625, 3.6 ± 0.03125 MHz etc. In order to prevent locking to any frequencies other than the colour subcarrier, the lock range of the PLL is restricted to half the line frequency, viz., 7.06 KHz.

After having recovered $Y$, $I$ and $Q$ signals, the $X$, $B$ and $G$ components are generated by using suitable proportions of $Y$, $I$ and $Q$ signals.

17.6 Frequency Synthesizer

The criteria for the design of a frequency synthesizer are the purity of the output waveform and the frequency stability. The important component units of a frequency synthesizer are frequency multipliers and dividers. A typical arrangement of a frequency synthesizer is shown in Fig. 17.6. Phase lock principles are utilized to realize frequency division and multiplication, thereby this helps achieving the desired end results. Typical arrangements for frequency multi-
Phase Lock Theory and Applications

Plotted in and division are shown in Fig. 17.7a and Fig. 17.7b respectively. In Fig. 17.7a, the output of the VCO is divided down by

\[ F\left(\frac{f_0}{f}\right) \]

Input

\[ \frac{f_0}{f}\]

VCO

\[ \frac{f_0}{f}\]

Fig. 17.7. Frequency divider and multiplication using phase lock techniques.

In frequency by the desired multiplication factor and the output is phase locked. Hence the output of the VCO gives the desired frequency multiplied version of the input wave. In Fig. 17.7b, where the output of the VCO gives the frequency division version of the input signal, the VCO output is multiplied by the required division factor and the multiplier output is phase locked to the reference signal. In these areas, digital PLLs could be gainfully utilized. In designing such devices, using PLL's, one has to take into consideration of the various sources of noise in loop viz., phase detector noise, multiplier noise, filter noise, amplitude noise, mixer noise, etc. This problem has been recently tackled by Kroupa [18].

17.7 mm-Wave Pulsed Coherent Signal

The generation of mm-wave pulsed coherent signals is done using the principles of injection synchronization and phase lock techniques [30,31] simultaneously. This is illustrated in Fig. 17.8. Here both
Gunn and IMPATT oscillators are used. Gunn oscillators are low power devices, but their internal noise is small compared to that of an IMPATT oscillators, which have higher power output. A judicious combination of them is sought in order to generate high power with low noise. The system, as shown in Fig. 17.8, operates in the following way. The output of the crystal oscillator is frequency multiplied with help of a PLL frequency multiplier. The output of the frequency multiplier is mixed in a harmonic mixer with the output of the Gunn oscillator to produce an i.f. signal, having the same frequency as that of the crystal oscillator. These two signals are then phase compared and the error signal, so obtained, controls the instantaneous frequency of the Gunn oscillator. The output of the CW Gunn oscillator, having forced to assume nearly the spectral character of the crystal oscillator, is used to injection synchronize a medium powered CW IMPATT oscillator. Injection synchronization is adopted for reasons of circuit simplicity. Finally, the output of the CW IMPATT oscillator is used to injection synchronize the high power pulsed IMPATT oscillator. The design consideration rests basically on the technique of narrowing the width of the output by the use of phase-lock principles.

17.8 Phased Array Antennas

The purpose of the phase array antenna (PAA) is to process the
received signal in such a way as to produce a resultant signal having a higher signal-to-noise ratio than the original signal. It consists of a number of independent antennas (say, N), usually spaced more than several wavelengths apart. If the noise components accompanying the signals from the group of N antennas are independent and the signal strengths are identical, then there will be a SNR gain of $(N)^{1/2}$ or $\sqrt{N}$ provided the signals are added coherently. Phase lock principle is utilized so that the signals add coherently. One of the ways of achieving this is to phase lock each of the antennas channels to the resultant signal obtained after the addition.

179 Digital Phase Shifting

The principle of injection phase locking can be utilized for digital phase shifting in microwave frequencies [33, 34]. At microwave frequencies digital phase shifting is carried out either with the use of discrete phase shifters for higher frequencies or with diode controlled phase shifters at lower frequencies. When the required phase shift is large, say 45°, it requires a number of the above mentioned phase shifters to be connected in cascade, and in such a case procedure leads to considerable cost and complexity.

Simplicity of the circuit arrangement of injection phase locking can be used to achieve the desired phase shift. To realize a phase shift of, say 45°, the frequency of the synchronizing signal is taken to be four times that of the oscillator frequency. The local oscillator is ultraharmonically locked to the synchronizing source. Now if a phase reversal is made for the synchronizing signal, the output phase of the local oscillator will change by $\pi/4$. Locking the local oscillator at one-fourth the frequency of the synchronizing may not be satisfactory. In such a case, two oscillators can be used in cascade, one operating at twice the frequency of the other. Thus the first oscillator is synchronized at half the frequency of the synchronizing input and the second oscillator is locked to the output to the first one operating at twice the frequency of the second oscillator. The design consideration should ensure stable locking.
Phase Lock Theories and Applications


<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(0)$</td>
<td>Normalized oscillator amplitude</td>
</tr>
<tr>
<td>$A(t)$</td>
<td>Oscillator amplitude</td>
</tr>
<tr>
<td>$A_r$</td>
<td>r.m.s. amplitude of the input signal</td>
</tr>
<tr>
<td>$A(t)$</td>
<td>Coefficients of Fokker-Planck equation</td>
</tr>
<tr>
<td>$\Delta P_{PLL}$</td>
<td>All digital phase locked loop</td>
</tr>
<tr>
<td>$B(t)$</td>
<td>Transformed value of $A(t)\cdot e^{j\Omega t}$</td>
</tr>
<tr>
<td>$B_P$</td>
<td>Bandwidth of bandpass filter</td>
</tr>
<tr>
<td>$B_L$</td>
<td>Bandpass limiter</td>
</tr>
<tr>
<td>$B_K$</td>
<td>Loop bandwidth</td>
</tr>
<tr>
<td>$B_N$</td>
<td>Noise bandwidth</td>
</tr>
<tr>
<td>$B_S$</td>
<td>Spectral bandwidth</td>
</tr>
<tr>
<td>$c$</td>
<td>Velocity of light</td>
</tr>
<tr>
<td>$C$</td>
<td>Capacitance</td>
</tr>
<tr>
<td>$CNR$</td>
<td>Carrier-to-noise ratio</td>
</tr>
<tr>
<td>$CNRT$</td>
<td>Threshold CNR</td>
</tr>
<tr>
<td>$DCO$</td>
<td>Digital controlled oscillator</td>
</tr>
<tr>
<td>$DPLL$</td>
<td>Digital phase locked loop</td>
</tr>
<tr>
<td>$D(f)$</td>
<td>Digital filter response function</td>
</tr>
<tr>
<td>$E$</td>
<td>Signal amplitude</td>
</tr>
<tr>
<td>$ERPLL$</td>
<td>Extended range phase locked loop</td>
</tr>
<tr>
<td>$E_C$</td>
<td>Statistical expectation operator</td>
</tr>
<tr>
<td>$ERDPLL$</td>
<td>Extended range digital phase locked loop</td>
</tr>
<tr>
<td>$FM$</td>
<td>Frequency modulation</td>
</tr>
<tr>
<td>$FP$</td>
<td>Fokker-Planck</td>
</tr>
<tr>
<td>$F_X(t)$</td>
<td>Fourier transform of $x(t)$</td>
</tr>
<tr>
<td>$F_0$</td>
<td>Transfer function of a filter</td>
</tr>
<tr>
<td>$P_D, P_s$</td>
<td>Ratio of the d.c. to a.c. gain of a filter</td>
</tr>
<tr>
<td>$FPFLO$</td>
<td>Frequency feedback phase locked oscillator</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Half power bandwidth</td>
</tr>
</tbody>
</table>
$g$ Acceleration due to gravity
$g'$ Noise and signal gain
$\gamma(n)$ Phase detector characteristic
$G_e$ Directive gain of a transmitter antenna
$G_r$ Directive gain of a receiver antenna
$h$ Planck's constant
$h(t)$ Impulse response of a time invariant network
$H(j)$ Loop transfer function
$I$ Value of an integral
$IF$ Intermediate frequency
$I_c$ Instantaneous value of the current through a capacitance
$I_0$ Instantaneous value of the current through an inductance
$I_{(X)}$ Instantaneous value of the plate current
$I_{(XY)}$ Modified Bessel function of order $n$ and argument $X$
$i_n(X)$ Modified Bessel function of the first kind and imaginary argument
$I_{SO}$ Injection synchronous oscillator
$k$ Boltzman's constant
$K$ One-sided lock range of an ISO or $K_pK_n$
$K_{r}$ r.m.s. amplitude of the VCO output
$K_p$ Phase detectors sensitivity
$K_n$ Sensitivity of the VCO
$L$ Inductance
$L_{(X)}$ Laplace transform of $x(t)$
$L_{dB}$ Transmission loss in dB
$LD$ Limiter discriminator
$m$ Modulation index
$m_{e}$ r.m.s. index of modulation
$M$ Modulation index error, Mutual inductance
$\nu_{(X)}$ Multifilar phase locked loop
$n$ Refractive index
$n(t)$ Bandpass noise
$N_{s}$ Single-sided noise power density
$N(t)$ Low pass noise with one sided power spectral density
$N_{s}$ Noise power output due to a series of noise impulses
$N_{s}$ Noise power output due to $N(t)$
$p$ d/dt, Heaviside operator
\( p(x) \) Probability density function
\( p(x | x_0, t) \) Probability distribution function
\( P_{x|y} (x | y) \) Conditional probability density
\( P_{y|z} (y | x) \) pdf of \( y(x) \) given \( y_0 = x(0) \)
\( p_{X, Y} (x, y) \) Joint probability density function of \( x \) and \( y \)
\( P(t) \) Period of the pulsed input
PAA Phase-array antenna
PD Phase detector
PiD Phase-locked demodulator
PLL Phase-locked loop
\( P(x | X) \) Cumulative probability density
PLR Phase locking range
PM Phase modulation
\( q(x, t) \) Random probability density function
\( q \) Instantaneous switch charge
\( Q(k) \) Initial period of the feedback signal of an ADPLL
\( q(y | z) \) Transition pdf of \( y(x) \) given \( y(z) = x \)
\( Q \) Quality factor of a tank circuit
\( Q_{out} \) Effective output of the oscillator
\( r_s \) Radial distance of the satellite from the observer
when passing overhead
\( R \) Resistance
\( R(\cdot) \) Autocorrelation function
\( R_{x,y}(\cdot) \) Crosscorrelation between \( x \) and \( y \)
\( S(\cdot) \) Spectral density function
\( SNR \) Signal-to-noise ratio
\( t \) Time variable
\( t \) Time variable
\( T_0 \) Time constant
\( T_1 \) Time constant
\( T \) Frequency acquisition time
\( T_{o/p} \) On/off frequency of the DCO or oscillator
\( T_{lock} \) Locking time
\( T_p \) Phase pulling time
\( T_c \) Expected time to slip cycle
VCO Voltage-controlled oscillator
\( w(\cdot) \) Steady state probability density function for the analog loop
\( W(\cdot) \) Steady state pdf for a DPLL
\( \cdot \) Overdot over a variable denotes time derivative
\( \ldots \) Denotes double differentiation with respect to time
\[ x \]
\[ \dot{x} \] Denotes \( \frac{dx}{dt} \) or a variable signifying space derivative.
\[ f'(x) \] Denotes \( \frac{df}{dx} \).
\[ f''(x) \] Denotes double differentiation with respect to space.
\( a \) Parameter to characterize \( w(\phi) \).
\( \nu \) Limiter signal suppression factor.
\( \gamma \) Parameter to characterize \( w(\phi) \).
\( \gamma_p \) Phase error overshoot.
\( y(\phi) \) Dirac delta function.
\( x^2 \) Lagrangian multiplier.
\( \zeta \) Damping factor.
\( \theta \) Angular deflection.
\( \theta(\phi) \) Angle modulation.
\( 0(\phi) \) Value of \( \theta(\phi) \) at the \( k \)th sampling instant.
\( \mu \) Amplification factor of a vacuum tube.
\( \sigma^2 \) Variance of the random variable \( x \).
\( \sigma^2(\phi) \) Phase variance at the \( k \)th sampling instant.
\( \sigma^2_p \) Phase error variance.
\( \sigma^2_n \) Oscillator amplitude fluctuation variance.
\( \sigma^2_e \) Phase error variance of the linearized DPLL.
\( \sigma^2_n \) Noise phase error variance in presence of signal modulation.
\( \Delta \) Modulation phase error variance.
\( \Delta \) Total phase error variance due to noise and signal.
\( \Delta_{\text{n}} \) Maximum frequency deviation.
\( \Delta_{\text{m}} \) Mean square frequency deviation.
\( \tau \) Normalized time \( (\phi) \).
\( r^e \) Correlation time.
\( \phi^e \) Instantaneous phase error.
\( \phi^e(\phi) \) Phase error at the \( k \)th sampling instant.
\( \phi^e_s \) Steady state phase error.
\( \phi_f \) Final value of phase error.
\( \phi_r \) Phase difference between the voltage and current in a tank circuit.
\( \xi(\phi) \) \( \phi \)-transform of \( \phi(\phi) \).
\( \phi(\phi) \) Instantaneous oscillator output phase.
\( \omega \)  \quad \text{Radian frequency}

\( \omega_0 \)  \quad \text{Loop natural frequency}

\( \omega_1 \)  \quad \text{Centre frequency of the voltage controlled oscillator, resonant frequency of the oscillator tank circuit}

\( \Omega \)  \quad \text{Angular frequency error}

\( \Omega_0 \)  \quad \text{Doppler frequency shift}

\( \Omega_1 \)  \quad \text{Upper side lock range}

\( \omega \)  \quad \text{Lower side lock range}
<table>
<thead>
<tr>
<th>INDEX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absorbing boundary, random walk problem 604-408</td>
</tr>
<tr>
<td>Acceleration application of Newton's law 37</td>
</tr>
<tr>
<td>due to gravity, pendulum equation 35</td>
</tr>
<tr>
<td>Accelerating news vehicle, effect on the received signal 188-190</td>
</tr>
<tr>
<td>Acquisition analysis 290-294, 305-308</td>
</tr>
<tr>
<td>analysis of Type-1 and Type-2 loops 260-268</td>
</tr>
<tr>
<td>approximate analysis of Type-2 loops 268-272</td>
</tr>
<tr>
<td>approximate analysis for PLL with an imperfect integrator 272-273</td>
</tr>
<tr>
<td>approximate analysis for a loop with perfect integrator 273-278</td>
</tr>
<tr>
<td>analysis, unified approach 278-283</td>
</tr>
<tr>
<td>time 171, 266</td>
</tr>
<tr>
<td>Table 328</td>
</tr>
<tr>
<td>Adaptive filter network 267</td>
</tr>
<tr>
<td>All digital phase-locked loop 433-418</td>
</tr>
<tr>
<td>Amplifying property 114-117, 157-155</td>
</tr>
<tr>
<td>Amplitude build-up of the oscillator output 59</td>
</tr>
<tr>
<td>distribution of the oscillator output 133</td>
</tr>
<tr>
<td>equations of the oscillator 58, 105, 120, 123</td>
</tr>
<tr>
<td>FM-PM conversion 162, 107</td>
</tr>
<tr>
<td>AM signal 105</td>
</tr>
<tr>
<td>effect on the oscillator output 105</td>
</tr>
<tr>
<td>effect on the oscillator phase 107</td>
</tr>
<tr>
<td>Amplifier multiplier 63</td>
</tr>
<tr>
<td>Angle modulated signal modulation 114-117</td>
</tr>
<tr>
<td>demodulation 480</td>
</tr>
<tr>
<td>effect on synchronization ratio 110</td>
</tr>
<tr>
<td>generation 482</td>
</tr>
<tr>
<td>sideband attenuation 110-114</td>
</tr>
</tbody>
</table>
Antenna aperture 216

Auto-stable multivibrator 166

Asymmetric frequency response of the band circuit 90, 92

property of the ISD 100

spatial distribution 99

Attenuation technique

taking nonlinear differential equation 54–57

application in the Pol equation 87–91

hard soft excited oscillator 65

soft self excited oscillator 61

Autoregressive function

bandpass noise 24

definition 15

relation with spectral density 16

white noise 25

Averaging value 14

Bandspain filter 355

Bandpass circuit 345–361

Bandwidth

band 34

noise 31, 237, 358

spectral 56

BPL

PLL preceded by 345–361

Cardiovascular noise ratio 238

Caruso’s rule of thumb 155, 240

Centre 40

Characteristic equation 41, 89

Chapman–Kolmogorov equation 459

Class A operation 33, 78

Class A 2 oscillator 78

Class Oscillation 33

Click

noise 429

state 435

Stability 428

Colour Identification in TV 454

Conditional probability density function 311

Conditional expectation 127
Continuous Markov process 390
Continuous random walk problems 357
Convolution 3, 5
Correlation function 16
Critical frequency 20
Critical point 2
Crystal controlled oscillator 167
Crystal filter 320
Crystal VCO 169
Cycle slipping phenomenon 494-408
Damping
critical 194
factor 193, 228
over 194
under 194
DC output of the PD 170
Delay time
effect on PLL equation 320
linear PLL response 322-322
non-linear behaviour of PLL 322-325
PLL stability 325-324
false acquisition 328-333
Demodulation 480
Deterministic signals 11
Differential equation 321
Digital controlled oscillator 440
Digital filter 440
Digital phase locked loop 439
Digital phase shifting 488
Diasor's delta function 6
Distortion
harmonics 209-211
intermodulation 209-211
self 209-211
third harmonic 65-62
Distortion-to-signal power ratio 116
Doppler shift 213
DPLL response to noisy signals 34-402
Duality 2
Bath noise temperature 218
Effective Q 1.1
Elliptic integral 76
Entertainment 83
Equation 13
Exception time to slip cycle 406
Expected value 13
Extended range
DPLL 423
phase locked demodulator 343
phase locked loop 292-294
Padding
slow 380
weak amplitude 363
Panning signal
Impedance channel 379
Rayleigh channel 378
Rayleigh channel 378
False acquisition 338-333
Phase locking 128
Filtering 483
Filtering property 112-114, 148-152
Filters
baseband 342
conventional 484
design 462-467
lowpass 148
lowpass equivalent of IF amplifier 313
optimisation 19, 231-235
PLL 583
First order
approximation 149
DPLL 444-449, 454
loop 194-408, 172-176
Flutter
frequency noise 252
phase noise 252
PM demodulator 237-249
Fuk tolerated equation circuit 791-794
for first order PLL 384-403
for ISO 126-128
steady state solution (ISO) 178
steady state solution, first order PLL 337-403
steady state solution, second order PLL 408-412

Fourier transform
definition 1
tables of 2

delta function 6

Frequency feedback phase locked loop 295-299

Frequency of oscillation
instantaneous 39
best angular 76
average value 82
roll-off 83

Frequency of skipping cycles 425

Frequency ramp 197, 452

Frequency response 50

Gain
directive 216
noise 321, 421
total 353, 421

Quasistatic
amplitude distribution 23
noise 22
oscillator amplitude 124
oscillator phase 129, 400

Hard-self-excited oscillator 63

Mayer's result 326

Heterodyne phase locked loop 310

Hopf
Wiener 20

Hydraulics 89

Input phase 44

Injection locking/synchronization 76

Equation synchronization PLL 299-303

Integration contour 256

Interference
filtering 123
weak 121
strong 124

Interfering tone 119

Effect on oscillator equation 120

Laplace transform 3
tables 4

Limit cycle 43, 53

stable 66
unstable 66
Limiting 33
Limiting, instantaneous/non-instantaneous 33
Limiting suppression factor 354
Linear approximation 179, 222
Linearized loop equation 182
Loop components 160-171
contax 481
equation (linearized) 187
first order 194-204, 384-403
phase locked 157
second order 268-288, 408-412
squaring 481
transfer function 184-188
type 1 116, 260
type 2 186, 260, 268, 275
Markovian process 387, 459
Mathematical analysis 171-179
Mean square loop error 229
Mean square phase error 130, 223, 253, 399, 455, 461
Mean square value 13
Mean square value of \( \dot{\theta} \) 402
Mean/expected time to skip-cycle 406
Minimum phase network 10
mm-wave pulsed coherent signal 486
Modulation
amplitude 102, 482
angle 107, 201-214, 217-249
correlation (variations) 418
FM/PM 682
Index 102, 107
Multipliers, analog 140
Main-tension VCO 169
Nurthy sinusoidal oscillations 33
Noise
amplifier 218
antenna 218
atmospheric 218
earth temperature 218
flicker frequency 252
flicker phase 252
glare and radio source 218
Gaussian 23
narrow band 24
502 - Phase-Lock Theories and Applications

thermal 23
white frequency 252
Nonlinear analysis (DPLL) 459-442
Non-symmetric spectral character 99
Optimization of lock performance 229-235
Optimum
ratio 211
phase locked demodulation 203
tracking 223

Oscillations
condenser discharge 24
hard self excited 65
mass-spring 34
mechanical vibrations 44
piezoelectric 34
resonant circuit 35
self-sustained 63
vibrations 49

Oscillator
crystal-controlled 167
L-C 167
response to AM signal 102
response to FM signal 107, 143
synchronization 78

Fundamental 35
period of oscillations 35

Phase detector
AWGN analysis 368

Phase-locked loop 76-101, 157-190

Phase-lock principle 76-101, 157-190
Spectral character, unlocked oscillator 97
Spectral density 16
Squaring loop 481
Stable focus 42
State space approach 37
Statistical linearization principles 164
Synchronization principles 78
Test-tone modulated signal 363
Third order loop 187
Threshold CNR 421
Time delay 320
PLL linear operation 321
PLL nonlinear behaviour 322
PLL stability 325
PLL acquisition range 337
Time scaling 2
Time shifting 2
Time to slip a cycle 406
Triangular phase detector 365, 374
Under damped system 200
Unlocked driven oscillator 97
Unstable focus 42
Unstable noise 42
van der Pol equation 49
Variance 14
Voltage control oscillator 166
White frequency noise 252
White phase noise 232
Yosts-Jackson formula 230
Z transform 469-473